Chapter
2

Measures

The last section of the previous chapter discusses several deficiencies of Riemann integration. To remedy those deficiencies, in this chapter we will extend the notion of the length of an interval to a larger collection of subsets of **R**. This will lead us to measures and then in the next chapter to integration with respect to measures.

We begin this chapter by investigatingf outer measure, which looks promising but fails to have a crucial property. That failure leads us to σ-algebras and measurable spaces. Then we define measures, in more abstract context that can be applied to settings more general than **R**. Next, we will construct Lebesgue measure on **R** as our desired extension of the notion of the length of an interval.

Fifth-century AD Roman ceiling mosaic in what is now a UNESCO World Heritage site in Ravenna, Italy. Giuseppe Vitali, who in 1905 proved result 2.17 in this chapter, was born and grew up in Ravenna, where he might have seen this mosaic. Could the memory of the translation-invariant feature of this mosaic have suggested to Vitali the translation invariance that is the heart of his proof of 2.17?

2A Outer Measure on $\mathbf{R}$

Motivation and Definition of Outer Measure

The Riemann integral arises from approximating the area under the graph of a function by sums of the areas of rectangles. These rectangles have heights that approximate values of the function on subintervals of the function's domain. The width of each approximating rectangle is the length of the corresponding subinterval. This length is the term $x_j - x_{j-1}$ in the definitions of lower and upper Riemann sums (see 1.3).

To extend integration to a larger class of functions than the Riemann integrable functions, we will write the domain of a function as the union of subsets more complicated than the subintervals used in Riemann integration. We will need to assign a size to each of those subsets, where the size is an extension of the length of intervals. For example, we expect the size of the set $(1,3) \cup (7,10)$ to be 5 (because the first interval has length 2, the second interval has length 3, and $2+3=5$).

Assigning a size to subsets of $\mathbf{R}$ that are more complicated than unions of open intervals becomes a nontrivial task. This chapter focuses on that task and its extension to other contexts. In the next chapter, we will see how to use the ideas developed in this chapter to create a rich theory of integration.

We begin by giving the expected definition of the length of an open interval, along with a notation for that length.

2.1 Definition ***length of open interval;*** $\ell(I)$

The *length* $\ell(I)$ of an open interval I is defined by

$$\ell(I) = \begin{cases} b-a & \text{if } I=(a,b) \text{ for some } a,b \in \mathbf{R} \text{ with } a<b, \\ 0 & \text{if } I=\varnothing, \\ \infty & \text{if } I=(-\infty,a) \text{ or } I=(a,\infty) \text{ for some } a \in \mathbf{R}, \\ \infty & \text{if } I=(-\infty,\infty). \end{cases}$$

Suppose $A \subset \mathbf{R}$. Every open set that contains A is the union of a sequence of open intervals (see Section D in the Appendix for a review of open sets, with special attention to 0.54). The size of A should be at most the sum of the lengths of those intervals. Taking the infimum of all such sums gives a reasonable definition of the size of A, denoted $|A|$ and called the outer measure of A.

2.2 Definition ***outer measure;*** $|A|$

The *outer measure* $|A|$ of a set $A \subset \mathbf{R}$ is defined by

$$|A| = \inf\Big\{\sum_{m=1}^{\infty} \ell(I_m) : I_1, I_2, \ldots \text{ are open intervals such that } A \subset \bigcup_{m=1}^{\infty} I_m\Big\}.$$

The definition of outer measure involves an infinite sum. The infinite sum $\sum_{m=1}^{\infty} t_m$ of a sequence $t_1, t_2, \ldots$ of elements of $[0,\infty]$ is defined to be ∞ if some $t_m = \infty$.

Otherwise, $\sum_{m=1}^{\infty} t_m$ is defined to be the supremum of the increasing sequence $t_1, t_1 + t_2, t_1 + t_2 + t_3, \ldots$ of partial sums.

2.3 Example *finite sets have outer measure* 0

Suppose $A = \{a_1, \ldots, a_N\}$ is a finite set of real numbers. Suppose $\varepsilon > 0$. Define a sequence $I_1, I_2, \ldots$ of open intervals by

$$I_m = \begin{cases} (a_m - \varepsilon, a_m + \varepsilon) & \text{if } m \leq N, \\ \varnothing & \text{if } m > N. \end{cases}$$

Then $I_1, I_2, \ldots$ is a sequence of open intervals whose union contains A. Clearly $\sum_{m=1}^{\infty} \ell(I_m) = 2\varepsilon N$. Hence $|A| \leq 2\varepsilon N$. Because ε is an arbitrary positive number, this implies that $|A| = 0$.

Good Properties of Outer Measure

Outer measure has several nice properties that will be discussed in this subsection. We begin with a result that improves upon the example above.

2.4 *Sequences have outer measure* 0

Every countable subset of **R** has outer measure 0.

Proof Suppose $A = \{a_1, a_2, \ldots\}$ is a countable subset of **R**. Let $\varepsilon > 0$. For $m \in \mathbf{Z}^+$, let

$$I_m = \Big(a_m - \frac{\varepsilon}{2^m}, a_m + \frac{\varepsilon}{2^m}\Big).$$

Then $I_1, I_2, \ldots$ is a sequence of open intervals whose union contains A. Because $\sum_{m=1}^{\infty} \ell(I_m) = 2\varepsilon$, we have $|A| \leq 2\varepsilon$. Because ε is an arbitrary positive number, this implies that $|A| = 0$. ■

The result above (along with 0.52) implies that the set **Q** of rational numbers has outer measure 0. We will soon show that there are far fewer rational numbers than real numbers (see 2.16). Thus the equation $|\mathbf{Q}| = 0$ indicates that outer measure has a good property that we want any reasonable notion of size to possess.

Greek stamp honoring mathematician Constantin Carathéodory, who proved important results about outer measure in the early twentieth century.

The next result shows that outer measure does the right thing with respect to set inclusion.

2.5 *Outer measure preserves order*

Suppose A and B are subsets of $\mathbf{R}$ with $A \subset B$. Then $|A| \leq |B|$.

Proof Suppose $I_1, I_2, \ldots$ is a sequence of open intervals whose union contains B. Then the union of this sequence of open intervals also contains A. Hence

$$|A| \leq \sum_{m=1}^{\infty} \ell(I_m).$$

Taking the infimum over all sequences of open intervals whose union contains B, we have $|A| \leq |B|$, as desired. ∎

We expect that the size of a subset of $\mathbf{R}$ should not change if the set is shifted to the right or to the left. The next definition will allow us to be more precise.

2.6 Definition ***translation;*** $t + A$

If $t \in \mathbf{R}$ and $A \subset \mathbf{R}$, then the *translation* $t + A$ is defined by

$$t + A = \{t + a : a \in A\}.$$

If $t > 0$, then $t + A$ is obtained by moving the set A to the right t units on the real line; if $t < 0$, then $t + A$ is obtained by moving the set A to the left $|t|$ units.

Translation does not change the length of an open interval. Specifically, if $t \in \mathbf{R}$ and $a, b \in [-\infty, \infty]$, then $t + (a, b) = (t + a, t + b)$ and thus $\ell\big(t + (a, b)\big) = \ell\big((a, b)\big)$. Here we are using the standard convention that $t + (-\infty) = -\infty$ and $t + \infty = \infty$.

The next result states that translation invariance carries over to outer measure.

2.7 *Outer measure is translation invariant*

Suppose $t \in \mathbf{R}$ and $A \subset \mathbf{R}$. Then $|t + A| = |A|$.

Proof Suppose $I_1, I_2, \ldots$ is a sequence of open intervals whose union contains A. Then $t + I_1, t + I_2, \ldots$ is a sequence of open intervals whose union contains $t + A$. Thus

$$|t + A| \leq \sum_{m=1}^{\infty} \ell(t + I_m) = \sum_{m=1}^{\infty} \ell(I_m).$$

Taking the infimum of the last term over all sequences $I_1, I_2, \ldots$ of open intervals whose union contains A, we have $|t + A| \leq |A|$.

Note that $A = -t + (t + A)$. Thus applying the inequality above with A replaced by $t + A$ and t replaced by $-t$, we have $|A| = |-t + (t + A)| \leq |t + A|$. Hence $|t + A| = |A|$, as desired. ∎

The union of the intervals $(1,4)$ and $(3,5)$ is the interval $(1,5)$. Thus

$$\ell\big((1,4)\cup(3,5)\big) < \ell\big((1,4)\big) + \ell\big((3,5)\big)$$

because the left side of the inequality above equals 4 and the right side equals 5. The direction of the inequality above is explained by noting that the interval $(3,4)$, which is the intersection of $(1,4)$ and $(3,5)$, has its length counted twice on the right side of the inequality above.

The example of the paragraph above should provide intuition for the direction of the inequality in the next result. The property of satisfying the inequality in the result below is called *countable subadditivity* because it applies to sequences of subsets.

2.8 *Countable subadditivity of outer measure*

Suppose $A_1, A_2, \dots$ is a sequence of subsets of $\mathbf{R}$. Then

$$\Big|\bigcup_{n=1}^{\infty} A_n\Big| \leq \sum_{n=1}^{\infty} |A_n|.$$

Proof Let $\varepsilon > 0$. For each $n \in \mathbf{Z}^+$, let $I_{1,n}, I_{2,n}, \dots$ be a sequence of open intervals whose union contains A_n such that

$$\sum_{m=1}^{\infty} \ell(I_{m,n}) \leq \frac{\varepsilon}{2^n} + |A_n|.$$

Thus

2.9
$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \ell(I_{m,n}) \leq \varepsilon + \sum_{n=1}^{\infty} |A_n|.$$

The doubly-indexed collection of open intervals $\{I_{m,n} : m,n \in \mathbf{Z}^+\}$ can be rearranged into a sequence of open intervals whose union contains $\bigcup_{n=1}^{\infty} A_n$ as follows, where in step k (start with $k = 2$, then $k = 3,4,5,\dots$) we adjoin the $k-1$ intervals whose indices add up to k:

$$\underbrace{I_{1,1}}_{2},\ \underbrace{I_{1,2}, I_{2,1}}_{3},\ \underbrace{I_{1,3}, I_{2,2}, I_{3,1}}_{4},\ \underbrace{I_{1,4}, I_{2,3}, I_{3,2}, I_{4,1}}_{5},\ \underbrace{I_{1,5}, I_{2,4}, I_{3,3}, I_{4,2}, I_{5,1}}_{\text{sum of the two indices equals 6}},\ \dots.$$

Inequality 2.9 shows that the sum of the lengths of the intervals listed above is less than or equal to $\varepsilon + \sum_{n=1}^{\infty}|A_n|$. Thus $\big|\bigcup_{n=1}^{\infty} A_n\big| \leq \varepsilon + \sum_{n=1}^{\infty}|A_n|$. Because ε is an arbitrary positive number, this implies that $\big|\bigcup_{n=1}^{\infty} A_n\big| \leq \sum_{n=1}^{\infty}|A_n|$, as desired. ∎

Countable subadditivity implies finite subadditivity, meaning that

$$|A_1 \cup \cdots \cup A_M| \leq |A_1| + \cdots + |A_M|$$

for all $A_1, \dots, A_M \subset \mathbf{R}$, because we can take $A_n = \varnothing$ for $n > M$ in 2.8.

The finite and countable subadditivity of outer measure, as proved above, add to our list of nice properties enjoyed by outer measure.

Outer Measure of a Closed Bounded Interval

One more good property of outer measure that we should prove is that if $a < b$, then the outer measure of the closed interval $[a, b]$ equals $b - a$. Indeed, if $\varepsilon > 0$, then $(a - \varepsilon, b + \varepsilon), \emptyset, \emptyset, \dots$ is a sequence of open intervals whose union contains $[a, b]$. Thus $|[a, b]| \leq b - a + 2\varepsilon$. Because this inequality holds for all $\varepsilon > 0$, we conclude that

$$|[a, b]| \leq b - a.$$

Is the inequality in the other direction obviously true to you? If so, think again, because a proof of the inequality in the other direction requires that completeness is used in some form. For example, suppose that **R** was a countable set (which is not true, as we will soon see, but the uncountability of **R** is not obvious). Then we would have $|[a, b]| = 0$ (by 2.4). Thus something deeper than you might suspect is going on with the ingredients needed to prove that $|[a, b]| \geq b - a$.

The following definition will be useful when we prove that $|[a, b]| \geq b - a$.

2.10 Definition *open cover*

Suppose $A \subset \mathbf{R}$.

- A collection $\mathcal{C}$ of open subsets of **R** is called an *open cover* of A if A is contained in the union of all the sets in $\mathcal{C}$.
- An open cover $\mathcal{C}$ of A is said to have a *finite subcover* if A is contained in the union of some finite list of sets in $\mathcal{C}$.

2.11 Example *open covers and finite subcovers*

- The collection $\{(n, n + 2) : n \in \mathbf{Z}^+\}$ is an open cover of $[2, 5]$ because $[2, 5] \subset \bigcup_{n=1}^{\infty} (n, n + 2)$. This open cover has a finite subcover because $[2, 5] \subset (1, 3) \cup (2, 4) \cup (3, 5) \cup (4, 6)$.
- The collection $\{(n, n + 2) : n \in \mathbf{Z}^+\}$ is an open cover of $[2, \infty)$ because $[2, \infty] \subset \bigcup_{n=1}^{\infty} (n, n + 2)$. This open cover does not have a finite subcover because there do not exist finitely many sets of the form $(n, n + 2)$ [with $n \in \mathbf{Z}^+$] whose union contains $[2, \infty)$.
- The collection $\{(0, 2 - \frac{1}{n}) : n \in \mathbf{Z}^+\}$ is an open cover of $(1, 2)$ because $(1, 2) \subset \bigcup_{n=1}^{\infty} (0, 2 - \frac{1}{n})$. This open cover does not have a finite subcover because there do not exist finitely many sets of the form $(0, 2 - \frac{1}{n})$ whose union contains $(1, 2)$.

The next result will be our major tool in the proof that $|[a, b]| \geq b - a$. Although we need only the result as stated, be sure to see Exercise 5 in this section, which when combined with the next result gives a characterization of the closed bounded subsets of **R**. Note that the following proof uses the completeness property of the real numbers (by asserting that the supremum of a certain nonempty bounded set exists).

See Section D in the Appendix for a review of closed sets.

2.12 *Heine–Borel Theorem*

Every open cover of a closed bounded subset of **R** has a finite subcover.

Proof Suppose E is a closed bounded subset of $\mathbf{R}$ and $\mathcal{C}$ is an open cover of E. We need to show that there exist finitely many sets $C_1, \dots, C_n \in \mathcal{C}$ such that $E \subset C_1 \cup \dots \cup C_n$.

First we consider the case where $E = [a, b]$ for some $a, b \in \mathbf{R}$ with $a < b$. Thus $\mathcal{C}$ is an open cover of $[a, b]$. Let

$$D = \{d \in [a, b] : [a, d] \text{ has a finite subcover from } \mathcal{C}\}.$$

Note that $a \in D$ (because $a \in C$ for some $C \in \mathcal{C}$). Thus D is not the empty set. Let

$$s = \sup D.$$

Thus $s \in [a, b]$. Hence there exists an open set $C \in \mathcal{C}$ such that $s \in C$. Let $\delta > 0$ be such that $(s - \delta, s + \delta) \subset C$. Because $s = \sup D$, there exist $d \in (s - \delta, s]$ and $n \in \mathbf{Z}^+$ and $C_1, \dots, C_n \in \mathcal{C}$ such that

$$[a, d] \subset C_1 \cup \dots \cup C_n.$$

Now

$$[a, d'] \subset C \cup C_1 \cup \dots \cup C_n \quad \text{for all } d' \in [s, s + \delta).$$

Thus $d' \in D$ for all $d' \in [s, s + \delta) \cap [a, b]$. This implies that $s = b$. Hence $b \in D$ (use the definitions of D and s to verify this), completing the proof in the case where $E = [a, b]$.

Now suppose E is an arbitrary closed bounded subset of $\mathbf{R}$ and that $\mathcal{C}$ is an open cover of E. Let $a, b \in \mathbf{R}$ be such that $E \subset [a, b]$. Now $\mathcal{C} \cup \{\mathbf{R} \setminus E\}$ is an open cover of $\mathbf{R}$ and hence is an open cover of $[a, b]$. By our first case, there exist $C_1, \dots, C_n \in \mathcal{C}$ such that

$$[a, b] \subset C_1 \cup \dots \cup C_n \cup (\mathbf{R} \setminus E).$$

Thus

$$E \subset C_1 \cup \dots \cup C_n,$$

completing the proof. ■

Saint-Affrique, the small town in southern France where Émile Borel (1871–1956) was born. Borel first stated and proved what we call the Heine–Borel Theorem in 1895. Earlier, German mathematician Eduard Heine (1821–1881) and others had used similar results.

Now we can prove that closed intervals have the expected outer measure.

2.13 *Outer measure of a closed interval*

Suppose $a, b \in \mathbf{R}$, with $a < b$. Then $|[a, b]| = b - a$.

Proof See the first paragraph of this subsection for the proof that $|[a, b]| \leq b - a$.

To prove the inequality in the other direction, suppose $I_1, I_2, \ldots$ is a sequence of open intervals such that $[a, b] \subset \bigcup_{m=1}^{\infty} I_m$. By the Heine–Borel Theorem (2.12), there exists $n \in \mathbf{Z}^+$ such that

2.14 $$[a, b] \subset I_1 \cup \cdots \cup I_n.$$

We will now prove by induction on n that the inclusion above implies that

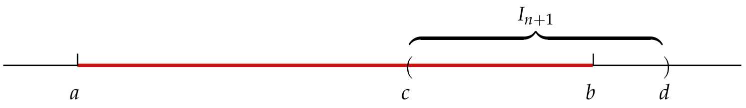
2.15 $$\sum_{m=1}^{n} \ell(I_m) \geq b - a.$$

This will then imply that $\sum_{m=1}^{\infty} \ell(I_m) \geq \sum_{m=1}^{n} \ell(I_m) \geq b - a$, completing the proof that $|[a, b]| \geq b - a$.

To get started with our induction, note that the 2.14 clearly implies 2.15 if $n = 1$. Now for the induction step: Suppose $n > 1$ and 2.14 implies 2.15 for all choices of $a, b \in \mathbf{R}$ with $a < b$. Suppose $I_1, \ldots, I_n, I_{n+1}$ are open intervals such that

$$[a, b] \subset I_1 \cup \cdots \cup I_n \cup I_{n+1}.$$

Thus b is in at least one of the intervals $I_1, \ldots, I_n, I_{n+1}$. By relabeling, we can assume that $b \in I_{n+1}$. Suppose $I_{n+1} = (c, d)$. If $c \leq a$, then $\ell(I_{n+1}) \geq b - a$ and there is nothing further to prove; thus we can assume that $a < c < b < d$, as shown in the figure below.



Hence

$$[a, c] \subset I_1 \cup \cdots \cup I_n.$$

By our induction hypothesis, $\sum_{m=1}^{n} \ell(I_m) \geq c - a$. Thus

$$\begin{aligned} \sum_{m=1}^{n+1} \ell(I_m) &\geq (c - a) + \ell(I_{n+1}) \\ &= (c - a) + (d - c) \\ &\geq b - a, \end{aligned}$$

completing the proof. ∎

The previous result has the following important corollary. You may be familiar with German mathematician Georg Cantor's (1845–1918) original proof of the next result. The proof using outer measure that is presented here gives an interesting alternative to Cantor's proof.

2.16 *Nontrivial intervals are uncountable*

Every interval in $\mathbf{R}$ that contains at least two distinct elements is uncountable.

Proof Suppose I is an interval that contains $a, b \in \mathbf{R}$ with $a < b$. Then

$$|I| \geq |[a, b]| = b - a > 0,$$

where the first inequality above holds because outer measure preserves order (see 2.5) and the equality above comes from 2.13. Because every countable subset of $\mathbf{R}$ has outer measure 0 (see 2.4), we can conclude that I is uncountable. ∎

Outer Measure is Not Additive

We have had several results giving nice properties of outer measure. Now we come to an unpleasant property of outer measure.

Outer measure led to the proof above that $\mathbf{R}$ is uncountable. This application of outer measure to prove a result that seems unconnected with outer measure is an indication that outer measure has serious mathematical value.

If outer measure was a perfect way to assign a size as an extension of the lengths of intervals, then the outer measure of the union of two disjoint sets would equal the sum of the outer measures of the two sets. Sadly, the next result states that outer measure does not have this property.

In the next section, we will begin the process of getting around the next result, which will lead us to measure theory.

2.17 *Nonadditivity of outer measure*

There exist disjoint subsets A and B of $\mathbf{R}$ such that

$$|A \cup B| \neq |A| + |B|.$$

Proof For $a \in [-1, 1]$, let E_a be the set of numbers in $[-1, 1]$ that differ from a by a rational number. In other words,

$$E_a = \{c \in [-1, 1] : a - c \in \mathbf{Q}\}.$$

If $a, b \in [-1, 1]$ and $E_a \cap E_b \neq \varnothing$, then $E_a = E_b$. (Proof: Suppose there exists $d \in E_a \cap E_b$. Then $a - d$ and $b - d$ are rational numbers; subtracting, we conclude that $a - b$ is a rational number. The equation $a - c = (a - b) + (b - c)$ now implies that if $c \in [-1, 1]$, then $a - c$ is a rational number if and only if $b - c$ is a rational number. In other words, $E_a = E_b$.)

Clearly $a \in E_a$ for each $a \in [-1, 1]$. Thus $[-1, 1] = \displaystyle\bigcup_{a \in [-1,1]} E_a$.

This step involves the Axiom of Choice, as discussed after this proof.

Let D be a set that contains exactly one element in each of the distinct sets in

$$\{E_a : a \in [-1,1]\}.$$

In other words, for every $a \in [-1,1]$, the set $D \cap E_a$ has exactly one element.

Let $r_1, r_2, \ldots$ be a sequence of distinct rational numbers such that

$$[-2,2] \cap \mathbf{Q} = \{r_1, r_2, \ldots\}.$$

Then

$$[-1,1] \subset \bigcup_{n=1}^{\infty} (r_n + D),$$

where the set inclusion above holds because if $a \in [-1,1]$, then $d \in E_a$ for some $d \in D$, which implies that $a - d \in \mathbf{Q}$, which implies that $a = r_n + d \in r_n + D$ for some $n \in \mathbf{Z}^+$.

The set inclusion above, the order preserving property of outer measure (2.5), and the countable subadditivity of outer measure (2.8) imply

$$|[-1,1]| \leq \sum_{n=1}^{\infty} |r_n + D|.$$

We know that $|[-1,1]| = 2$ (from 2.13). The translation invariance of outer measure (2.7) thus allows us to rewrite the inequality above as

$$2 \leq \sum_{n=1}^{\infty} |D|.$$

Thus $|D| > 0$.

Note that the sets $r_1 + D, r_2 + D, \ldots$ are disjoint. (Proof: Suppose there exists $t \in (r_m + D) \cap (r_n + D)$. Then $t = r_m + d_1 = r_n + d_2$ for some $d_1, d_2 \in D$, which implies that $d_1 - d_2 = r_n - r_m \in \mathbf{Q}$. Our construction of D now implies that $d_1 = d_2$, which implies that $r_m = r_n$, which implies that $m = n$.)

Let $M \in \mathbf{Z}^+$ be such that $M > \frac{6}{|D|}$. Clearly

$$\bigcup_{n=1}^{M} (r_n + D) \subset [-3,3]$$

because $D \subset [-1,1]$ and each $r_n \in [-2,2]$. The set inclusion above implies that

$$\left| \bigcup_{n=1}^{M} (r_n + D) \right| \leq 6.$$

However

$$\sum_{n=1}^{M} |r_n + D| = \sum_{n=1}^{M} |D| = M\,|D| > 6.$$

Thus

2.18
$$\left|\bigcup_{n=1}^{M}(r_n + D)\right| < \sum_{n=1}^{M}|r_n + D|.$$

If we had $|A \cup B| = |A| + |B|$ for all disjoints subsets A, B of $\mathbf{R}$, then by induction on M we would have $\left|\bigcup_{n=1}^{M} A_n\right| = \sum_{n=1}^{M}|A_n|$ for all disjoint subsets $A_1, \dots, A_M$ of $\mathbf{R}$. However, 2.18 tells us that no such result holds. Thus there exist disjoint subsets A, B of $\mathbf{R}$ such that $|A \cup B| \neq |A| + |B|$. ∎

The Axiom of Choice, which belongs to set theory, states that if $\mathcal{E}$ is a set whose elements are disjoint sets, then there exists a set D that contains exactly one element in each set that is an element of $\mathcal{E}$. We used the Axiom of Choice to construct the set D that was used in the last proof.

A small minority of mathematicians objects to the use of the Axiom of Choice. Thus we will keep track of where we need to use it. Even if you do not like to use the Axiom of Choice, the previous result warns us away from trying to prove that outer measure is additive (any such proof would need to contradict the Axiom of Choice, which is consistent with the standard axioms of set theory).

EXERCISES 2A

1 Suppose A and B are subsets of $\mathbf{R}$ and $|B| = 0$. Prove that $|A \cup B| = |A|$.

2 Suppose $A \subset \mathbf{R}$ and $t \in \mathbf{R}$. Let $tA = \{ta : a \in A\}$. Prove that $|tA| = |t|\,|A|$.

3 Suppose $A, B \subset \mathbf{R}$ and $|A| < \infty$. Prove that $|B \setminus A| \geq |B| - |A|$.

4 Suppose $A \subset \mathbf{R}$. Prove that $|A| = \lim_{n\to\infty}|A \cap [-n, n]|$.

5 Suppose E is a subset of $\mathbf{R}$ with the property that every open cover of E has a finite subcover. Prove that E is closed and bounded.

6 Suppose $\mathcal{A}$ is a set of closed subsets of $\mathbf{R}$ such that $\bigcap_{A\in\mathcal{A}} A = \varnothing$. Prove that if $\mathcal{A}$ contains at least one bounded set, then there exist $M \in \mathbf{Z}^+$ and $A_1, \dots, A_M \in \mathcal{A}$ such that $A_1 \cap \dots \cap A_M = \varnothing$.

7 Suppose $a, b \in \mathbf{R}$ with $a < b$. Prove that
$$|(a, b)| = |[a, b)| = |(a, b]| = b - a.$$

8 Suppose a, b, c, d are real numbers with $a < b$ and $c < d$. Prove that
$$|(a, b) \cup (c, d)| = (b - a) + (d - c) \text{ if and only if } (a, b) \cap (c, d) = \varnothing.$$

9 Suppose $I_1, I_2, \dots$ is a disjoint sequence of open intervals. Prove that
$$\left|\bigcup_{n=1}^{\infty} I_n\right| = \sum_{n=1}^{\infty} \ell(I_n).$$

10 Prove that $|[0,1] \setminus \mathbf{Q}| = 1$.

11 Suppose $r_1, r_2, \ldots$ is a sequence that contains every rational number. Let

$$A = \mathbf{R} \setminus \bigcup_{n=1}^{\infty} \Bigl(r_n - \frac{1}{2^n}, r_n + \frac{1}{2^n}\Bigr).$$

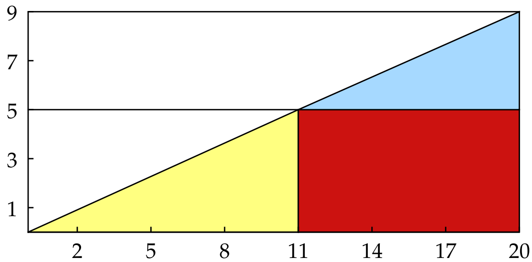
(a) Show that A is a closed subset of $\mathbf{R}$.

(b) Prove that if I is an interval contained in A, then I contains at most one element.

(c) Prove that $|A| = \infty$.

12 Suppose $\varepsilon > 0$. Prove that there exists a subset A of $[0,1]$ such that A is closed, every element of A is an irrational number, and $|A| > 1 - \varepsilon$.

13 Consider the following figure, which is drawn accurately to scale.



(a) Show that the right triangle whose vertices are $(0,0)$, $(20,0)$, and $(20,9)$ has area 90.
[*We have not defined area yet, but just use the elementary formulas for the areas of triangles and rectangles that you learned long ago.*]

(b) Show that the yellow right triangle has area 27.5.

(c) Show that the red rectangle has area 45.

(d) Show that the blue right triangle has area 18.

(e) Add the results of parts (b), (c), and (d), showing that the area of the colored region is 90.5.

(f) Seeing the figure above, most people expect that parts (a) and (e) will have the same result. Yet in part (a) we found area 90, and in part (e) we found area 90.5. Explain why these results differ.
[*You may be tempted to think that what we have here is a two-dimensional example similar to the result about the nonadditivity of outer measure* (2.17). *However, examples of nonadditivity require much more complicated sets than in this example.*]

2B Measurable Spaces and Functions

The last result in the previous section showed that outer measure is not additive. Perhaps this disappointing result could be fixed by using some notion other than outer measure for the size of a subset of $\mathbf{R}$? However, the next result shows that there does not exist a notion of size [called the Greek letter mu (μ) in the result below] that has all the desirable properties.

The property in the third bullet point below is called *countable additivity*. Countable additivity is a highly desirable property because we want to be able to prove theorems about limits (the heart of analysis!), which requires countable additivity.

2.19 *Nonexistence of extension of length to all subsets of* R

There does not exist a function μ with all the following properties:

- μ is a function from the set of subsets of $\mathbf{R}$ to $[0, \infty]$;
- $\mu(I) = \ell(I)$ for every open interval I of $\mathbf{R}$;
- $\mu\Big(\bigcup_{n=1}^{\infty} A_n\Big) = \sum_{n=1}^{\infty} \mu(A_n)$ for every disjoint sequence $A_1, A_2, \dots$ of subsets of $\mathbf{R}$;
- $\mu(t + A) = \mu(A)$ for every subset A of $\mathbf{R}$ and every $t \in \mathbf{R}$.

This will be a proof by contradiction, as is common with nonexistence proofs.
We will show that μ has all the properties of outer measure that were used in the proof of 2.17.

Proof Suppose that there exists a function μ with all the properties listed in the statement of this result.

Observe that $\mu(\emptyset) = 0$ (this follows from the second bullet point because the empty set is an open interval with length 0).

If $A \subset B \subset \mathbf{R}$, then $\mu(A) \leq \mu(B)$ (this follows from the third bullet point because we can write B as the union of the disjoint sequence $A, B \setminus A, \emptyset, \emptyset, \dots$; thus $\mu(B) = \mu(A) + \mu(B \setminus A) + 0 + 0 + \cdots = \mu(A) + \mu(B \setminus A) \geq \mu(A)$).

If $a, b \in \mathbf{R}$ with $a < b$, then $(a, b) \subset [a, b] \subset (a - \varepsilon, b + \varepsilon)$ for every $\varepsilon > 0$. Thus $b - a \leq \mu([a, b]) \leq b - a + 2\varepsilon$ for every $\varepsilon > 0$. Hence $\mu([a, b]) = b - a$.

If $A_1, A_2, \dots$ is a sequence of subsets of $\mathbf{R}$, then $A_1, A_2 \setminus A_1, A_3 \setminus (A_1 \cup A_2), \dots$ is a disjoint sequence of subsets of $\mathbf{R}$ whose union equals $\bigcup_{n=1}^{\infty} A_n$. Thus

$$\begin{aligned}
\mu\Big(\bigcup_{n=1}^{\infty} A_n\Big) &= \mu\Big(A_1 \cup (A_2 \setminus A_1) \cup \big(A_3 \setminus (A_1 \cup A_2)\big) \cup \cdots\Big) \\
&= \mu(A_1) + \mu(A_2 \setminus A_1) + \mu\big(A_3 \setminus (A_1 \cup A_2)\big) + \cdots \\
&\leq \sum_{n=1}^{\infty} \mu(A_n),
\end{aligned}$$

where the second equality follows from the countable additivity of μ.

We have shown that μ has all the properties of outer measure that were used in the proof of 2.17. Repeating the proof of 2.17, we see that there exist disjoint subsets A, B of $\mathbf{R}$ such that $\mu(A \cup B) \neq \mu(A) + \mu(B)$. Thus the disjoint sequence $A, B, \emptyset, \emptyset, \ldots$ does not satisfy the countable additivity property required by the third bullet point. This contradiction completes the proof. ■

σ-Algebras

The last result shows that we need to give up one of the desirable properties in our goal of extending the notion of size from intervals to more general subsets of $\mathbf{R}$. We cannot give up the second bullet point in 2.19 because the size of an interval needs to be its length. We cannot give up the third bullet point because countable additivity is needed to prove theorems about limits . We cannot give up the fourth bullet point because a size that is not translation invariant does not satisfy our intuitive notion of size as a generalization of length.

Thus we are forced to relax the requirement in the first bullet point of 2.19 that the size is defined for all subsets of $\mathbf{R}$. Experience shows that to have a viable theory that allows for taking limits, the collection of subsets for which the size is defined should be closed under complementation and closed under countable unions. Thus we make the following definition.

2.20 Definition ***σ-algebra***

Suppose X is a set and $\mathcal{S}$ is a set of subsets of X. Then $\mathcal{S}$ is called a *σ-algebra* on X if the following three conditions are satisfied:

- $\emptyset \in \mathcal{S}$;
- if $E \in \mathcal{S}$, then $X \setminus E \in \mathcal{S}$;
- if $E_1, E_2, \ldots$ is a sequence of elements of $\mathcal{S}$, then $\displaystyle\bigcup_{n=1}^{\infty} E_n \in \mathcal{S}$.

Make sure you verify that the examples in all three bullet points below are indeed σ-algebras. The verification is obvious for the first two bullet points. For the third bullet point, you need to use the result that the countable union of countable sets is countable (see the proof of 2.8 for an example of how a doubly-indexed list can be converted to a singly-indexed sequence). The exercises contain some additional examples of σ-algebras.

2.21 Example *σ-algebras*

- Suppose X is a set. Then clearly $\{\emptyset, X\}$ is a σ-algebra on X.
- Suppose X is a set. Then clearly the set of all subsets of X is a σ-algebra on X.
- Suppose X is a set. Then the set of all subsets E of X such that E is countable or $X \setminus E$ is countable is a σ-algebra on X.

Now we come to some easy but important properties of σ-algebras.

2.22 ***σ-algebras are closed under countable intersection***

Suppose $\mathcal{S}$ is a σ-algebra on some set X. Then

- $X \in \mathcal{S}$;
- if $D, E \in \mathcal{S}$, then $D \cup E \in \mathcal{S}$ and $D \cap E \in \mathcal{S}$ and $D \setminus E \in \mathcal{S}$;
- if $E_1, E_2, \ldots$ is a sequence of elements of $\mathcal{S}$, then $\displaystyle\bigcap_{n=1}^{\infty} E_n \in \mathcal{S}$.

Proof Because $\varnothing \in \mathcal{S}$ and $X = X \setminus \varnothing$, the first two bullet points of the definition of σ-algebra imply that $X \in \mathcal{S}$.

Suppose $D, E \in \mathcal{S}$. Then $D \cup E$ equals the union of the sequence $D, E, \varnothing, \varnothing, \ldots$ of elements of $\mathcal{S}$. Thus the third bullet point in the definition of σ-algebra implies that $D \cup E \in \mathcal{S}$.

De Morgan's Laws (0.58) tell us that

$$X \setminus (D \cap E) = (X \setminus D) \cup (X \setminus E).$$

If $D, E \in \mathcal{S}$, then the right side of the equation above is in $\mathcal{S}$; hence $X \setminus (D \cap E) \in \mathcal{S}$; thus the complement in X of $X \setminus (D \cap E)$ is in $\mathcal{S}$; in other words, $D \cap E \in \mathcal{S}$.

Because $D \setminus E = D \cap (X \setminus E)$, we see that if $D, E \in \mathcal{S}$, then $D \setminus E \in \mathcal{S}$.

Finally, suppose $E_1, E_2, \ldots$ is a sequence of elements of $\mathcal{S}$. De Morgan's Laws (0.58) tell us that

$$X \setminus \bigcap_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} (X \setminus E_n).$$

The right side of the equation above is in $\mathcal{S}$. Hence the left side is in $\mathcal{S}$, which implies that $X \setminus (X \setminus \bigcap_{n=1}^{\infty} E_n) \in \mathcal{S}$. In other words, $\bigcap_{n=1}^{\infty} E_n \in \mathcal{S}$. ∎

The word *measurable* is used in the terminology introduced below because in the next section we will introduce a size function, called a measure, defined on measurable sets.

2.23 Definition ***measurable space; measurable set***

- A *measurable space* is an ordered pair $(X, \mathcal{S})$, where X is a set and $\mathcal{S}$ is a σ-algebra on X.
- An element of $\mathcal{S}$ is called an *$\mathcal{S}$-measurable set*, or just a *measurable set* if $\mathcal{S}$ is clear from the context.

For example, if $X = \mathbf{R}$ and $\mathcal{S}$ equals the set of all subsets of $\mathbf{R}$ that are countable or have a countable complement, then the set of rational numbers is $\mathcal{S}$-measurable but the set of positive real numbers is not $\mathcal{S}$-measurable.

Borel Subsets of $\mathbf{R}$

The next result guarantees that there is a smallest σ-algebra on a set X containing a given set $\mathcal{A}$ of subsets of X.

2.24 *Smallest σ-algebra containing a collection of subsets*

Suppose X is a set and $\mathcal{A}$ is a set of subsets of X. Then the intersection of all σ-algebras on X that contain $\mathcal{A}$ is a σ-algebra on X.

Proof There is at least one σ-algebra on X that contains $\mathcal{A}$ because the σ-algebra consisting of all subsets of X contains $\mathcal{A}$.

Let $\mathcal{S}$ equal the intersection of all σ-algebras on X that contain $\mathcal{A}$. Then $\varnothing \in \mathcal{S}$ because $\varnothing$ is an element of each σ-algebra on X that contains $\mathcal{A}$.

Suppose $E \in \mathcal{S}$. Thus E is in every σ-algebra on X that contains $\mathcal{A}$. Thus $X \setminus E$ is in every σ-algebra on X that contains $\mathcal{A}$. Hence $X \setminus E \in \mathcal{S}$.

Suppose $E_1, E_2, \dots$ is a sequence of elements of $\mathcal{S}$. Thus each E_n is in every σ-algebra on X that contains $\mathcal{A}$. Thus $\bigcup_{n=1}^{\infty} E_n$ is in every σ-algebra on X that contains $\mathcal{A}$. Hence $\bigcup_{n=1}^{\infty} E_n \in \mathcal{S}$, which completes the proof that $\mathcal{S}$ is a σ-algebra on X. ∎

Using the terminology *smallest* for the intersection of all σ-algebras that contain a set $\mathcal{A}$ of subsets of X makes sense because the intersection of those σ-algebras is contained in every σ-algebra that contains $\mathcal{A}$.

2.25 Example *smallest σ-algebra*

- Suppose X is a set and $\mathcal{A}$ is the set of subsets of X that consist of exactly one element:
$$\mathcal{A} = \{\{x\} : x \in X\}.$$
Then the smallest σ-algebra on X containing $\mathcal{A}$ is the set of all subsets E of X such that E is countable or $X \setminus E$ is countable, as you should verify.

- Suppose $\mathcal{A} = \{(0,1), (0,\infty)\}$. Then the smallest σ-algebra on $\mathbf{R}$ containing $\mathcal{A}$ is $\{\varnothing, (0,1), (0,\infty), (-\infty,0] \cup [1,\infty), (-\infty,0], [1,\infty), (-\infty,1), \mathbf{R}\}$, as you should verify.

Now we come to a crucial definition.

2.26 Definition *Borel set*

A *Borel set* (also called *a Borel subset of* $\mathbf{R}$) is an element of the smallest σ-algebra on $\mathbf{R}$ containing all open subsets of $\mathbf{R}$.

We have defined the Borel subsets of $\mathbf{R}$ to be the smallest σ-algebra on $\mathbf{R}$ containing all the open subsets of $\mathbf{R}$. We could have defined the Borel subsets of $\mathbf{R}$ to be the smallest σ-algebra on $\mathbf{R}$ containing all the open intervals (because every open subset of $\mathbf{R}$ is the union of a sequence of open intervals—see 0.54).

2.27 Example *Borel sets*

- Every closed subset of $\mathbf{R}$ is a Borel set because every closed subset of $\mathbf{R}$ is the complement of an open subset of $\mathbf{R}$.
- Every countable subset of $\mathbf{R}$ is a Borel set because if $B = \{x_1, x_2, \dots\}$, then $B = \bigcup_{n=1}^{\infty} \{x_n\}$, which is a Borel set because each $\{x_n\}$ is a closed set.
- Every half-open interval $[a, b)$ (where $a, b \in \mathbf{R}$) is a Borel set because $[a, b) = \bigcap_{n=1}^{\infty} (a - \frac{1}{n}, b)$.
- If $f \colon \mathbf{R} \to \mathbf{R}$ is a function, then the set of points at which f is continuous is the intersection of a sequence of open sets (see Exercise 12 in this section) and thus is a Borel set.

The intersection of every sequence of open subsets of $\mathbf{R}$ is a Borel set. However, the set of all such intersections is not equal to the set of Borel sets (because it is not closed under countable unions). The set of all countable unions of countable intersections of open subsets of $\mathbf{R}$ is also not equal to the set of Borel sets (because it is not closed under countable intersections). And so on—there is no concrete procedure for constructing the collection of Borel sets.

We will see later that there exist subsets of $\mathbf{R}$ that are not Borel sets. However, any subset of $\mathbf{R}$ that you can write down in a concrete fashion will be a Borel set.

Inverse Images

The next definition will be used frequently in the rest of this chapter.

2.28 Definition ***inverse image; $f^{-1}(A)$***

If $f \colon X \to Y$ is a function and $A \subset Y$, then the set $f^{-1}(A)$ is defined by

$$f^{-1}(A) = \{x \in X : f(x) \in A\}.$$

2.29 Example *inverse images*

Suppose $f \colon [0, 4\pi] \to \mathbf{R}$ is defined by $f(x) = \sin x$. Then

$$\begin{aligned} f^{-1}\big((0, \infty)\big) &= (0, \pi) \cup (2\pi, 3\pi), \\ f^{-1}([0, 1]) &= [0, \pi] \cup [2\pi, 3\pi] \cup \{4\pi\}, \\ f^{-1}(\{-1\}) &= \{\tfrac{3\pi}{2}, \tfrac{7\pi}{2}\}, \\ f^{-1}\big((2, 3)\big) &= \varnothing, \end{aligned}$$

as you should verify.

Inverse images have good algebraic properties, as is shown in the next two results.

2.30 *The algebra of inverse images*

Suppose $f\colon X \to Y$ is a function. Then

- $f^{-1}(Y \setminus A) = X \setminus f^{-1}(A)$ for every $A \subset Y$;
- $f^{-1}(\bigcup_{A \in \mathcal{A}} A) = \bigcup_{A \in \mathcal{A}} f^{-1}(A)$ for every set $\mathcal{A}$ of subsets of Y;
- $f^{-1}(\bigcap_{A \in \mathcal{A}} A) = \bigcap_{A \in \mathcal{A}} f^{-1}(A)$ for every set $\mathcal{A}$ of subsets of Y.

Proof Suppose $A \subset Y$. For $x \in X$ we have

$$\begin{aligned} x \in f^{-1}(Y \setminus A) &\iff f(x) \in Y \setminus A \\ &\iff f(x) \notin A \\ &\iff x \notin f^{-1}(A) \\ &\iff x \in X \setminus f^{-1}(A). \end{aligned}$$

Thus $f^{-1}(Y \setminus A) = X \setminus f^{-1}(A)$, which proves the first bullet point.

To prove the second bullet point, suppose $\mathcal{A}$ is a set of subsets of Y. Then

$$\begin{aligned} x \in f^{-1}\Big(\bigcup_{A \in \mathcal{A}} A\Big) &\iff f(x) \in \bigcup_{A \in \mathcal{A}} A \\ &\iff f(x) \in A \text{ for some } A \in \mathcal{A} \\ &\iff x \in f^{-1}(A) \text{ for some } A \in \mathcal{A} \\ &\iff x \in \bigcup_{A \in \mathcal{A}} f^{-1}(A). \end{aligned}$$

Thus $f^{-1}(\bigcup_{A \in \mathcal{A}} A) = \bigcup_{A \in \mathcal{A}} f^{-1}(A)$, which proves the second bullet point.

The third bullet point is proved in the same fashion as the second bullet point, with unions replaced by intersections and *for some* replaced by *for every*. ∎

2.31 *Inverse image of a composition*

Suppose $f\colon X \to Y$ and $g\colon Y \to W$ are functions. Then

$$(g \circ f)^{-1}(A) = f^{-1}\big(g^{-1}(A)\big)$$

for every $A \subset W$.

Proof Suppose $A \subset W$. For $x \in X$ we have

$$\begin{aligned} x \in (g \circ f)^{-1}(A) \iff (g \circ f)(x) \in A &\iff g\big(f(x)\big) \in A \\ &\iff f(x) \in g^{-1}(A) \\ &\iff x \in f^{-1}\big(g^{-1}(A)\big). \end{aligned}$$

Thus $(g \circ f)^{-1}(A) = f^{-1}\big(g^{-1}(A)\big)$, as desired. ∎

Measurable Functions

The next definition tells us which real-valued functions behave reasonably with respect to a σ-algebra on their domain.

2.32 Definition ***measurable function***

Suppose $(X, \mathcal{S})$ is a measurable space. A function $f : X \to \mathbf{R}$ is called *$\mathcal{S}$-measurable* (or just *measurable* if $\mathcal{S}$ is clear from the context) if

$$f^{-1}(B) \in \mathcal{S}$$

for every Borel set $B \subset \mathbf{R}$.

2.33 Example *measurable functions*

- If $\mathcal{S} = \{\emptyset, X\}$, then the only $\mathcal{S}$-measurable functions from X to $\mathbf{R}$ are the constant functions.
- If $\mathcal{S}$ is the set of all subsets of X, then every function from X to $\mathbf{R}$ is $\mathcal{S}$-measurable.
- If $\mathcal{S} = \{\emptyset, (-\infty, 0), [0, \infty), \mathbf{R}\}$ (which is a σ-algebra on $\mathbf{R}$), then a function $f : \mathbf{R} \to \mathbf{R}$ is $\mathcal{S}$-measurable if and only if f is constant on $(-\infty, 0)$ and f is constant on $[0, \infty)$.

Another class of examples comes from characteristic functions, which are defined below. The Greek letter chi (χ) is traditionally used to denote a characteristic function.

2.34 Definition ***characteristic function;*** χ_E

Suppose E is a subset of a set X. The *characteristic function* of E is the function $\chi_E : X \to \mathbf{R}$ defined by

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E, \\ 0 & \text{otherwise.} \end{cases}$$

The set X that contains E is not explicitly included in the notation χ_E because X will always be clear from the context.

2.35 Example *inverse image with respect to a characteristic function*

Suppose $(X, \mathcal{S})$ is a measurable space, $E \subset X$, and $B \subset \mathbf{R}$. Then

$$\chi_E{}^{-1}(B) = \begin{cases} E & \text{if } 0 \notin B \text{ and } 1 \in B, \\ X \setminus E & \text{if } 0 \in B \text{ and } 1 \notin B, \\ X & \text{if } 0 \in B \text{ and } 1 \in B, \\ \emptyset & \text{if } 0 \notin B \text{ and } 1 \notin B. \end{cases}$$

Thus we see that χ_E is an $\mathcal{S}$-measurable function if and only if $E \in \mathcal{S}$.

Note that if $f \colon X \to \mathbf{R}$ is a function and $a \in \mathbf{R}$, then

$$f^{-1}\big((a,\infty)\big) = \{x \in X : f(x) > a\}.$$

The definition of an $\mathcal{S}$-measurable function requires that the inverse image of every Borel subset of $\mathbf{R}$ is in $\mathcal{S}$. The next result shows that to verify that a function is $\mathcal{S}$-measurable, we can check the inverse images of a much smaller collection of subsets of $\mathbf{R}$.

2.36 *Condition for measurable function*

Suppose $(X,\mathcal{S})$ is a measurable space and $f \colon X \to \mathbf{R}$ is a function such that

$$f^{-1}\big((a,\infty)\big) \in \mathcal{S}$$

for all $a \in \mathbf{R}$. Then f is an $\mathcal{S}$-measurable function.

Proof Let

$$\mathcal{T} = \{A \subset \mathbf{R} : f^{-1}(A) \in \mathcal{S}\}.$$

We want to show that every Borel subset of $\mathbf{R}$ is in $\mathcal{T}$. To do this, we will first show that $\mathcal{T}$ is a σ-algebra on $\mathbf{R}$.

Certainly $\varnothing \in \mathcal{T}$, because $f^{-1}(\varnothing) = \varnothing \in \mathcal{S}$.

If $A \in \mathcal{T}$, then $f^{-1}(A) \in \mathcal{S}$; hence (by 2.30)

$$f^{-1}(\mathbf{R} \setminus A) = X \setminus f^{-1}(A) \in \mathcal{S}$$

and thus $\mathbf{R} \setminus \mathcal{A} \in \mathcal{T}$. In other words, $\mathcal{T}$ is closed under complementation.

If $A_1, A_2, \ldots \in \mathcal{T}$, then $f^{-1}(A_1), f^{-1}(A_2), \ldots \in \mathcal{S}$; hence (by 2.30)

$$f^{-1}\Big(\bigcup_{n=1}^{\infty} A_n\Big) = \bigcup_{n=1}^{\infty} f^{-1}(A_n) \in \mathcal{S}$$

and thus $\bigcup_{n=1}^{\infty} A_n \in \mathcal{T}$. In other words, $\mathcal{T}$ is closed under countable unions. Thus $\mathcal{T}$ is a σ-algebra on $\mathbf{R}$.

By hypothesis, $\mathcal{T}$ contains $\{(a,\infty) : a \in \mathbf{R}\}$. Because $\mathcal{T}$ is closed under complementation, $\mathcal{T}$ also contains $\{(-\infty,b] : b \in \mathbf{R}\}$. Because the σ-algebra $\mathcal{T}$ is closed under finite intersections (by 2.22), we see that $\mathcal{T}$ contains $\{(a,b] : a,b \in \mathbf{R}\}$. Because $(a,b) = \bigcup_{n=1}^{\infty}(a, b - \frac{1}{n}]$ and $(-\infty,b) = \bigcup_{n=1}^{\infty}(-n, b - \frac{1}{n}]$ and $\mathcal{T}$ is closed under countable unions, we can conclude that $\mathcal{T}$ contains every open subset of $\mathbf{R}$ (use 0.54).

Thus the σ-algebra $\mathcal{T}$ contains the smallest σ-algebra on $\mathbf{R}$ that contains all open subsets of $\mathbf{R}$. In other words, $\mathcal{T}$ contains every Borel subset of $\mathbf{R}$. Thus f is an $\mathcal{S}$-measurable function. ∎

In the result above, we could replace the collection of sets $\{(a,\infty) : a \in \mathbf{R}\}$ by any collection of subsets of $\mathbf{R}$ such that the smallest σ-algebra containing that collection contains the Borel subsets of $\mathbf{R}$. For specific examples of such collections of subsets of $\mathbf{R}$, see Exercises 3–6.

We have been dealing with $\mathcal{S}$-measurable functions from X to $\mathbf{R}$ in the context of an arbitrary set X and a σ-algebra $\mathcal{S}$ on X. An important special case of this setup is when X is a Borel subset of $\mathbf{R}$ and $\mathcal{S}$ is the set of Borel subsets of $\mathbf{R}$ that are contained in X (see Exercise 11 for another way of thinking about this σ-algebra). In this special case, the $\mathcal{S}$-measurable functions are called Borel measurable.

2.37 Definition ***Borel measurable function***

Suppose $X \subset \mathbf{R}$. A function $f\colon X \to \mathbf{R}$ is called *Borel measurable* if $f^{-1}(B)$ is a Borel set for every Borel set $B \subset \mathbf{R}$.

If $X \subset \mathbf{R}$ and there exists a Borel measurable function $f\colon X \to \mathbf{R}$, then X must be a Borel set [because $X = f^{-1}(\mathbf{R})$].

If $X \subset \mathbf{R}$ and $f\colon X \to \mathbf{R}$ is a function, then f is a Borel measurable function if and only if $f^{-1}\big((a,\infty)\big)$ is a Borel set for every $a \in \mathbf{R}$ (use 2.36).

Suppose X is a set and $f\colon X \to \mathbf{R}$ is a function. The measurability of f depends upon the choice of a σ-algebra on X. If the σ-algebra is called $\mathcal{S}$, then we can discuss whether f is an $\mathcal{S}$-measurable function. If X is a Borel subset of $\mathbf{R}$, then $\mathcal{S}$ might equal the set of Borel sets contained in X, in which case the phrase *Borel measurable* means the same as $\mathcal{S}$-measurable. However, whether or not $\mathcal{S}$ is a collection of Borel sets, we consider inverse images of Borel subsets of $\mathbf{R}$ when determining whether a function is $\mathcal{S}$-measurable.

The next result states that continuity interacts well with the notion of Borel measurability.

2.38 ***Every continuous function is Borel measurable***

Every continuous real-valued function defined on a Borel subset of $\mathbf{R}$ is a Borel measurable function.

Proof Suppose $X \subset \mathbf{R}$ is a Borel set and $f\colon X \to \mathbf{R}$ is continuous. To prove that f is Borel measurable, fix $a \in \mathbf{R}$.

If $x \in X$ and $f(x) > a$, then (by the continuity of f) there exists $\delta_x > 0$ such that $f(y) > a$ for all $y \in (x - \delta_x, x + \delta_x) \cap X$. Thus

$$f^{-1}\big((a,\infty)\big) = \Bigg(\bigcup_{x \in f^{-1}((a,\infty))} (x - \delta_x, x + \delta_x)\Bigg) \cap X.$$

The union inside the large parentheses above is an open subset of $\mathbf{R}$ (by 0.50), and hence its intersection with X is a Borel set. Thus we can conclude that $f^{-1}\big((a,\infty)\big)$ is a Borel set.

Now 2.36 implies that f is a Borel measurable function. ■

Now we come to another class of Borel measurable functions. A similar definition could be made for decreasing functions, with a corresponding similar result.

2.39 Definition ***increasing function***

Suppose $X \subset \mathbf{R}$ and $f \colon X \to \mathbf{R}$ is a function.

- f is called *increasing* if $f(x) \le f(y)$ for all $x, y \in X$ with $x < y$.
- f is called *strictly increasing* if $f(x) < f(y)$ for all $x, y \in X$ with $x < y$.

2.40 ***Every increasing function is Borel measurable***

Every increasing function defined on a Borel subset of $\mathbf{R}$ is a Borel measurable function.

Proof Suppose $X \subset \mathbf{R}$ is a Borel set and $f \colon X \to \mathbf{R}$ is increasing. To prove that f is Borel measurable, fix $a \in \mathbf{R}$.

Let $b = \inf f^{-1}\big((a, \infty)\big)$. Then it is easy to see that

$$f^{-1}\big((a,\infty)\big) = (b, \infty) \cap X \quad \text{or} \quad f^{-1}\big((a,\infty)\big) = [b, \infty) \cap X.$$

Either way, we can conclude that $f^{-1}\big((a,\infty)\big)$ is a Borel set.

Now 2.36 implies that f is a Borel measurable function. ■

The next result shows that measurability interacts well with composition.

2.41 ***Composition of measurable functions***

Suppose $(X, \mathcal{S})$ is a measurable space and $f \colon X \to \mathbf{R}$ is an $\mathcal{S}$-measurable function. Suppose g is a Borel measurable function defined on a subset of $\mathbf{R}$ that includes the range of f. Then $g \circ f \colon X \to \mathbf{R}$ is an $\mathcal{S}$-measurable function.

Proof Suppose $B \subset \mathbf{R}$ is a Borel set. Then (see 2.31)

$$(g \circ f)^{-1}(B) = f^{-1}\big(g^{-1}(B)\big).$$

Because g is a Borel measurable function, $g^{-1}(B)$ is a Borel subset of $\mathbf{R}$. Because f is an $\mathcal{S}$-measurable function, $f^{-1}\big(g^{-1}(B)\big) \in \mathcal{S}$. Thus the equation above implies that $(g \circ f)^{-1}(B) \in \mathcal{S}$. Thus $g \circ f$ is an $\mathcal{S}$-measurable function. ■

2.42 Example *if f is measurable, then so are $-f, \frac{1}{2}f, |f|, f^2$*

Suppose $(X, \mathcal{S})$ is a measurable space and $f \colon X \to \mathbf{R}$ is an $\mathcal{S}$-measurable function. Then the functions $-f, \frac{1}{2}f, |f|, f^2$ are all $\mathcal{S}$-measurable functions because each of these functions can be written as the composition with a continuous (and thus Borel measurable) function g, and then using the result above.

Specifically, take $g(x) = -x$, then $g(x) = \frac{1}{2}x$, then $g(x) = |x|$, and then $g(x) = x^2$.

Measurability also interacts well with algebraic operations, as shown in the next result.

2.43 *Algebraic operations with measurable functions*

Suppose $(X, \mathcal{S})$ is a measurable space and $f, g \colon X \to \mathbf{R}$ are $\mathcal{S}$-measurable. Then

- $f + g$, $f - g$, and fg are $\mathcal{S}$-measurable functions;
- if $g(x) \neq 0$ for all $x \in X$, then $\frac{f}{g}$ is an $\mathcal{S}$-measurable function.

Proof Let $r_1, r_2, \dots$ be a sequence of rational numbers that contains every rational number.

Suppose $a \in \mathbf{R}$. We will show that

2.44 $$(f+g)^{-1}\big((a,\infty)\big) = \bigcup_{n=1}^{\infty} \Big(f^{-1}\big((r_n,\infty)\big) \cap g^{-1}\big((a-r_n,\infty)\big) \Big),$$

which implies that $(f+g)^{-1}\big((a,\infty)\big) \in \mathcal{S}$.

To prove 2.44, first suppose $x \in (f+g)^{-1}\big((a,\infty)\big)$. Thus $f(x) + g(x) > a$. Hence the open interval $\big(a - g(x), f(x)\big)$ is nonempty, and thus it contains some rational number r_n. This implies that $f(x) > r_n$ [which means that $x \in f^{-1}\big((r_n,\infty)\big)$] and $r_n > a - g(x)$ [which implies that $x \in g^{-1}\big((a-r_n,\infty)\big)$]. Thus x is an element of the right side of 2.44, completing the proof that the left side of 2.44 is contained in the right side.

The proof of the inclusion in the other direction is easier. Specifically, suppose $x \in f^{-1}\big((r_n,\infty)\big) \cap g^{-1}\big((a-r_n,\infty)\big)$ for some n. Thus

$$f(x) > r_n \quad \text{and} \quad g(x) > a - r_n.$$

Adding these two inequalities, we see that $f(x) + g(x) > a$. Thus x is an element of the left side of 2.44, completing the proof of 2.44. Hence $f + g$ is an $\mathcal{S}$-measurable function.

Example 2.42 tells us that $-g$ is an $\mathcal{S}$-measurable function. Thus $f - g$, which equals $f + (-g)$ is an $\mathcal{S}$-measurable function.

The easiest way to prove that fg is an $\mathcal{S}$-measurable function uses the equation

$$fg = \frac{(f+g)^2 - f^2 - g^2}{2}.$$

The operation of squaring an $\mathcal{S}$-measurable function produces an $\mathcal{S}$-measurable function (see Example 2.42), as does the operation of multiplication by $\frac{1}{2}$ (again, see Example 2.42). Thus the equation above implies that fg is an $\mathcal{S}$-measurable function.

Suppose $g(x) \neq 0$ for all $x \in X$. The function defined on $\mathbf{R} \setminus \{0\}$ (a Borel subset of $\mathbf{R}$) that takes x to $\frac{1}{x}$ is continuous and thus is a Borel measurable function (by 2.38). Now 2.41 implies that $\frac{1}{g}$ is an $\mathcal{S}$-measurable function. Combining this result with what we have already proved about the product of $\mathcal{S}$-measurable functions, we conclude that $\frac{f}{g}$ is an $\mathcal{S}$-measurable function. ∎

The next result shows that the pointwise limit of a sequence of $\mathcal{S}$-measurable functions is $\mathcal{S}$-measurable. This is a highly desirable property (recall that the set of Riemann integrable functions on some interval is not closed under taking pointwise limits; see Example 1.17).

2.45 ***Limit of $\mathcal{S}$-measurable functions***

Suppose $(X, \mathcal{S})$ is a measurable space and $f_1, f_2, \dots$ is a sequence of $\mathcal{S}$-measurable functions from X to $\mathbf{R}$. Suppose $\lim_{n\to\infty} f_n(x)$ exists for each $x \in X$. Define $f\colon X \to \mathbf{R}$ by

$$f(x) = \lim_{n\to\infty} f_n(x).$$

Then f is an $\mathcal{S}$-measurable function.

Proof Suppose $a \in \mathbf{R}$. We will show that

2.46 $$f^{-1}\big((a,\infty)\big) = \bigcup_{k=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} f_n^{-1}\big((a+\tfrac{1}{k},\infty)\big),$$

which implies that $f^{-1}\big((a,\infty)\big) \in \mathcal{S}$.

To prove 2.46, first suppose $x \in f^{-1}\big((a,\infty)\big)$. Thus there exists $k \in \mathbf{Z}^+$ such that $f(x) > a + \frac{1}{k}$. The definition of limit now implies that there exists $m \in \mathbf{Z}^+$ such that $f_n(x) > a + \frac{1}{k}$ for all $n \geq m$. Thus x is in the right side of 2.46, proving that the left side of 2.46 is contained in the right side.

To prove the inclusion in the other direction, suppose x is in the right side of 2.46. Thus there exist $k, m \in \mathbf{Z}^+$ such that $f_n(x) > a + \frac{1}{k}$ for all $n \geq m$. Taking the limit as $n \to \infty$, we see that $f(x) \geq a + \frac{1}{k} > a$. Thus x is in the left side of 2.46, completing the proof of 2.46. Thus f is an $\mathcal{S}$-measurable function. ∎

Occasionally we need to consider functions that take values in $[-\infty,\infty]$. For example, even if we start with a sequence of real-valued functions in 2.50, we might end up with functions with values in $[-\infty,\infty]$. Thus we extend the notion of Borel sets to subsets of $[-\infty,\infty]$, as follows.

2.47 Definition ***Borel subsets of $[-\infty,\infty]$***

A subset of $[-\infty,\infty]$ is called a *Borel set* if its intersection with $\mathbf{R}$ is a Borel set.

In other words, a set $C \subset [-\infty,\infty]$ is a Borel set if and only if there exists a Borel set $B \subset \mathbf{R}$ such that $C = B$ or $C = B \cup \{\infty\}$ or $C = B \cup \{-\infty\}$ or $C = B \cup \{\infty, -\infty\}$.

You should verify that with the definition above, the set of Borel subsets of $[-\infty,\infty]$ is a σ-algebra on $[-\infty,\infty]$.

Next, we extend the definition of $\mathcal{S}$-measurable functions to functions taking values in $[-\infty,\infty]$.

2.48 Definition ***measurable function***

Suppose $(X, \mathcal{S})$ is a measurable space. A function $f\colon X \to [-\infty, \infty]$ is called *$\mathcal{S}$-measurable* if

$$f^{-1}(B) \in \mathcal{S}$$

for every Borel set $B \subset [-\infty, \infty]$.

The next result, which is analogous to 2.36, states that we need not consider all Borel subsets of $[-\infty, \infty]$ when taking inverse images to determine whether or not a function with values in $[-\infty, \infty]$ is $\mathcal{S}$-measurable.

2.49 ***Condition for measurable function***

Suppose $(X, \mathcal{S})$ is a measurable space and $f\colon X \to [-\infty, \infty]$ is a function such that

$$f^{-1}\big((a, \infty]\big) \in \mathcal{S}$$

for all $a \in \mathbf{R}$. Then f is an $\mathcal{S}$-measurable function.

The proof of the result above is left to the reader (also see Exercise 23 in this section).

We end this section by showing that the pointwise infimum and pointwise supremum of a sequence of $\mathcal{S}$-measurable functions is $\mathcal{S}$-measurable.

2.50 ***Infimum and supremum of a sequence of $\mathcal{S}$-measurable functions***

Suppose $(X, \mathcal{S})$ is a measurable space and $f_1, f_2, \dots$ is a sequence of $\mathcal{S}$-measurable functions from X to $[-\infty, \infty]$. Define $g, h\colon X \to [-\infty, \infty]$ by

$$g(x) = \inf\{f_n(x) : n \in \mathbf{Z}^+\} \quad \text{and} \quad h(x) = \sup\{f_n(x) : n \in \mathbf{Z}^+\}.$$

Then g and h are $\mathcal{S}$-measurable functions.

Proof Let $a \in \mathbf{R}$. The definition of the supremum implies that

$$h^{-1}\big((a, \infty]\big) = \bigcup_{n=1}^{\infty} f_n{}^{-1}\big((a, \infty]\big),$$

as you should verify. The equation above, along with 2.49, implies that h is an $\mathcal{S}$-measurable function.

Note that

$$g(x) = -\sup\{-f_n(x) : n \in \mathbf{Z}^+\}$$

for all $x \in X$. Thus the result about the supremum implies that g is an $\mathcal{S}$-measurable function. ∎

EXERCISES 2B

1 Show that $\mathcal{S} = \{\bigcup_{n \in K}(n, n+1] : K \subset \mathbf{Z}\}$ is a σ-algebra on $\mathbf{R}$.

2 Verify both bullet points in Example 2.25.

3 Suppose $\mathcal{S}$ is the smallest σ-algebra on $\mathbf{R}$ containing $\{(r, s] : r, s \in \mathbf{Q}\}$. Prove that $\mathcal{S}$ equals the collection of Borel subsets of $\mathbf{R}$.

4 Suppose $\mathcal{S}$ is the smallest σ-algebra on $\mathbf{R}$ containing $\{(r, n] : r \in \mathbf{Q}, n \in \mathbf{Z}\}$. Prove that $\mathcal{S}$ equals the collection of Borel subsets of $\mathbf{R}$.

5 Suppose $\mathcal{S}$ is the smallest σ-algebra on $\mathbf{R}$ containing $\{(r, r+1) : r \in \mathbf{Q}\}$. Prove that $\mathcal{S}$ equals the collection of Borel subsets of $\mathbf{R}$.

6 Suppose $\mathcal{S}$ is the smallest σ-algebra on $\mathbf{R}$ containing $\{[r, \infty) : r \in \mathbf{Q}\}$. Prove that $\mathcal{S}$ equals the collection of Borel subsets of $\mathbf{R}$.

7 Prove that the collection of Borel subsets of $\mathbf{R}$ is translation invariant. More precisely, prove that if $B \subset \mathbf{R}$ is a Borel set and $t \in \mathbf{R}$, then $t + B$ is a Borel set.

8 Prove that the collection of Borel subsets of $\mathbf{R}$ is dilation invariant. More precisely, prove that if $B \subset \mathbf{R}$ is a Borel set and $t \in \mathbf{R}$, then tB (which is defined to be $\{tb : b \in B\}$) is a Borel set.

9 Give an example of a measurable space $(X, \mathcal{S})$ and a function $f\colon X \to \mathbf{R}$ such that $|f|$ is $\mathcal{S}$-measurable but f is not $\mathcal{S}$-measurable.

10 Show that the set of real numbers that have a decimal expansion with the digit 5 appearing infinitely often is a Borel set.

11 Suppose $\mathcal{T}$ is a σ-algebra on a set Y and $X \in \mathcal{T}$. Let $\mathcal{S} = \{E \in \mathcal{T} : E \subset X\}$.

(a) Show that $\mathcal{S} = \{F \cap X : F \in \mathcal{T}\}$.

(b) Show that $\mathcal{S}$ is a σ-algebra on X.

12 Suppose $f\colon \mathbf{R} \to \mathbf{R}$ is a function.

(a) For $n \in \mathbf{Z}^+$, let A_n equal

$$\{a \in \mathbf{R} : |f(b) - f(c)| < \tfrac{1}{n} \text{ for all } b, c \in (a - \delta, a + \delta) \text{ for some } \delta > 0\}.$$

Prove that A_n is an open subset of $\mathbf{R}$ for each $n \in \mathbf{Z}^+$.

(b) Prove that the set of points at which f is continuous equals $\bigcap_{n=1}^{\infty} A_n$.

(c) Conclude that the set of points at which f is continuous is a Borel set.

13 Suppose $(X, \mathcal{S})$ is a measurable space, $E_1, \dots, E_n$ are disjoint subsets of X, and $c_1, \dots, c_n$ are distinct real numbers. Prove that $c_1 \chi_{E_1} + \cdots + c_n \chi_{E_n}$ is an $\mathcal{S}$-measurable function if and only if $E_1, \dots, E_n \in \mathcal{S}$.

14 Suppose $f_1, f_2, \ldots$ is a sequence of functions from a set X to $\mathbf{R}$. Explain why

$$\{x \in X : \text{the sequence } f_1(x), f_2(x), \ldots \text{ has a limit in } \mathbf{R}\}$$
$$= \bigcap_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{m=N}^{\infty} (f_N - f_m)^{-1}\bigl((-\tfrac{1}{k}, \tfrac{1}{k})\bigr).$$

15 Suppose $(X, \mathcal{S})$ is a measurable space and $f_1, f_2, \ldots$ is a sequence of $\mathcal{S}$-measurable functions from X to $\mathbf{R}$. Prove that

$$\{x \in X : \text{the sequence } f_1(x), f_2(x), \ldots \text{ has a limit in } \mathbf{R}\}$$

is an $\mathcal{S}$-measurable subset of X.

16 Suppose X is a set and $E_1, E_2, \ldots$ is a disjoint sequence of subsets of X such that $\bigcup_{n=1}^{\infty} E_n = X$. Let $\mathcal{S} = \{\bigcup_{n \in K} E_n : K \subset \mathbf{Z}^+\}$.

(a) Show that $\mathcal{S}$ is a σ-algebra on X.

(b) Prove that a function from X to $\mathbf{R}$ is $\mathcal{S}$-measurable if and only if the function is constant on E_n for every $n \in \mathbf{Z}^+$.

17 Suppose $\mathcal{S}$ is a σ-algebra of subsets of some set X. Suppose $A \subset X$. Let

$$\mathcal{S}_A = \{E \in \mathcal{S} : A \subset E \text{ or } A \cap E = \varnothing\}.$$

(a) Prove that $\mathcal{S}_A$ is a σ-algebra of subsets of X.

(b) Suppose $f\colon X \to \mathbf{R}$ is a function. Prove that f is measurable with respect to $\mathcal{S}_A$ if and only if f is measurable with respect to $\mathcal{S}$ and f is constant on A.

18 Suppose X is a Borel subset of $\mathbf{R}$ and $f\colon X \to \mathbf{R}$ is a function such that $\{x \in X : f \text{ is not continuous at } x\}$ is a countable set. Prove f is a Borel measurable function.

19 Suppose $f\colon \mathbf{R} \to \mathbf{R}$ is differentiable at every element of $\mathbf{R}$. Prove that f' is a Borel measurable function from $\mathbf{R}$ to $\mathbf{R}$.

20 Suppose X is a nonempty set and $\mathcal{S}$ is the σ-algebra on X consisting of all subsets of X that are either countable or have a countable complement in X. Give a characterization of the $\mathcal{S}$-measurable real-valued functions on X.

21 Suppose $(X, \mathcal{S})$ is a measurable space and $f, g\colon X \to \mathbf{R}$ are $\mathcal{S}$-measurable functions. Prove that if $f(x) > 0$ for all $x \in X$, then f^g (which is the function whose value at $x \in X$ equals $f(x)^{g(x)}$) is an $\mathcal{S}$-measurable function.

22 Prove 2.49.

23 Prove or give a counterexample: If $(X, \mathcal{S})$ is a measurable space and

$$f\colon X \to [-\infty, \infty]$$

is a function such that $f^{-1}\bigl((a, \infty)\bigr) \in \mathcal{S}$ for every $a \in \mathbf{R}$, then f is an $\mathcal{S}$-measurable function.

24 Suppose $B \subset \mathbf{R}$ and $f: B \to \mathbf{R}$ is an increasing function. Prove that f is continuous at every element of B except for a countable subset of B.

25 Suppose $B \subset \mathbf{R}$ and $f: B \to \mathbf{R}$ is a strictly increasing function. Prove that $f^{-1}: f(B) \to \mathbf{R}$ is a continuous function.

26 Suppose $B \subset \mathbf{R}$ is $f: B \to \mathbf{R}$ is an increasing function. Prove that if $A \subset B$ is a Borel set, then $f(A)$ is a Borel set.

27 Suppose $B \subset \mathbf{R}$ and $f: B \to \mathbf{R}$ is an increasing function. Prove that there exists a sequence $f_1, f_2, \ldots$ of strictly increasing functions from B to $\mathbf{R}$ such that

$$f(x) = \lim_{n\to\infty} f_n(x)$$

for every $x \in B$.

28 Suppose $B \subset \mathbf{R}$ and $f: B \to \mathbf{R}$ is a bounded increasing function. Prove that there exists an increasing function $g: \mathbf{R} \to \mathbf{R}$ such that $g(x) = f(x)$ for all $x \in B$.

29 Suppose $f: B \to \mathbf{R}$ is a Borel measurable function. Define $g: \mathbf{R} \to \mathbf{R}$ by

$$g(x) = \begin{cases} f(x) & \text{if } x \in B, \\ 0 & \text{if } x \in \mathbf{R} \setminus B. \end{cases}$$

Prove that g is a Borel measurable function.

2C Measures and Their Properties

Definition and Examples of Measures

The original motivation for the next definition came from trying to extend the notion of the length of an interval. However, the definition below allows us to discuss size in many more contexts. For example, we will see later that the area of a set in the plane or the volume of a set in higher dimensions fits into this structure. The word *measure* allows us to use a single word instead of repeating theorems for *length*, *area*, and *volume*.

2.51 Definition ***measure***

Suppose X is a set and $\mathcal{S}$ is a σ-algebra on X. A *measure* on $(X, \mathcal{S})$ is a function $\mu\colon \mathcal{S} \to [0, \infty]$ such that $\mu(\varnothing) = 0$ and

$$\mu\Big(\bigcup_{n=1}^{\infty} E_n\Big) = \sum_{n=1}^{\infty} \mu(E_n)$$

for every disjoint sequence $E_1, E_2, \dots$ of sets in $\mathcal{S}$.

The countable additivity that forms the key part of the definition above will allow us to prove good limit theorems. Note that countable additivity implies finite additivity: if μ is a measure on $(X, \mathcal{S})$ and $E_1, \dots, E_M$ are disjoint sets in $\mathcal{S}$, then

$$\mu(E_1 \cup \cdots \cup E_M) = \mu(E_1) + \cdots + \mu(E_M),$$

as follows from applying the equation $\mu(\varnothing) = 0$ and countable additivity to the disjoint sequence $E_1, \dots, E_M, \varnothing, \varnothing, \dots$ of sets in $\mathcal{S}$.

In the mathematical literature, sometimes a measure on $(X, \mathcal{S})$ is just called a measure on X if the σ-algebra $\mathcal{S}$ is clear from the context.

The concept of a measure, as defined here, is sometimes called a positive measure *(although the phrase* nonnegative measure *would be more accurate).*

2.52 Example *measures*

- If X is a set, then *counting measure* is the measure μ defined on the σ-algebra of all subsets of X by setting $\mu(E) = n$ if E is a finite set containing exactly n elements and $\mu(E) = \infty$ if E is not a finite set.
- Suppose X is a set, $\mathcal{S}$ is a σ-algebra on X, and $c \in X$. Define the *Dirac* measure δ_c on $(X, \mathcal{S})$ by

$$\delta_c(E) = \begin{cases} 1 & \text{if } c \in E, \\ 0 & \text{if } c \notin E. \end{cases}$$

  This measure is named in honor of the British mathematician and physicist Paul Dirac (1902–1984), who won the Nobel Prize for Physics in 1933 for his work combining relativity and quantum mechanics at the atomic level.

- Suppose X is a set, $\mathcal{S}$ is a σ-algebra on X, and $w\colon X \to [0,\infty]$ is a function. Define a measure μ on $(X,\mathcal{S})$ by
$$\mu(E) = \sum_{x\in E} w(x)$$
for $E \in \mathcal{S}$. [Here the sum is defined as the supremum of all finite subsums $\sum_{x\in D} w(x)$ as D ranges over all finite subsets of E.]

- Suppose X is a set and $\mathcal{S}$ is the σ-algebra on X consisting of all subsets of X that are either countable or have a countable complement in X. Define a measure μ on $(X,\mathcal{S})$ by
$$\mu(E) = \begin{cases} 0 & \text{if } E \text{ is countable,} \\ 3 & \text{if } E \text{ is uncountable.} \end{cases}$$

- Suppose $\mathcal{S}$ is the σ-algebra on $\mathbf{R}$ consisting of all subsets of $\mathbf{R}$. Then the function that takes a set $E \subset \mathbf{R}$ to $|E|$ (the outer measure of E) is not a measure because it is not finitely additive (see 2.17).

- Suppose $\mathcal{B}$ is the σ-algebra on $\mathbf{R}$ consisting of all Borel subsets of $\mathbf{R}$. Then we will see in the next section that outer measure is a measure on $(\mathbf{R},\mathcal{B})$.

The following terminology is frequently useful.

2.53 Definition ***measure space***

A *measure space* is an ordered triple $(X,\mathcal{S},\mu)$, where X is a set, $\mathcal{S}$ is a σ-algebra on X, and μ is a measure on $(X,\mathcal{S})$.

Properties of Measures

The hypothesis that $\mu(D) < \infty$ is needed in the second bullet point of the next result to avoid undefined expressions of the form $\infty - \infty$.

2.54 ***Measure preserves order; measure of a set difference***

Suppose $(X,\mathcal{S},\mu)$ is a measure space and $D, E \in \mathcal{S}$ are such that $D \subset E$. Then

- $\mu(D) \le \mu(E)$;
- $\mu(E \setminus D) = \mu(E) - \mu(D)$ provided that $\mu(D) < \infty$.

Proof Because $E = D \cup (E \setminus D)$ and this is a disjoint union, we have
$$\mu(E) = \mu(D) + \mu(E \setminus D) \ge \mu(D),$$
which proves the first bullet point. If $\mu(D) < \infty$, then subtracting $\mu(D)$ from both sides of the equation above proves the second bullet point. ∎

The countable additivity property of measures applies to disjoint countable unions. The following countable subadditivity property applies to countable unions that may not be disjoint unions.

2.55 *Countable subadditivity*

Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $E_1, E_2, \ldots \in \mathcal{S}$. Then

$$\mu\Bigl(\bigcup_{n=1}^{\infty} E_n\Bigr) \leq \sum_{n=1}^{\infty} \mu(E_n).$$

Proof Let $D_1 = \varnothing$ and $D_n = E_1 \cup \cdots \cup E_{n-1}$ for $n \geq 2$. Then

$$E_1 \setminus D_1, E_2 \setminus D_2, E_3 \setminus D_3, \ldots$$

is a disjoint sequence of subsets of X whose union equals $\bigcup_{n=1}^{\infty} E_n$. Thus

$$\begin{aligned}
\mu\Bigl(\bigcup_{n=1}^{\infty} E_n\Bigr) &= \mu\Bigl(\bigcup_{n=1}^{\infty} (E_n \setminus D_n)\Bigr) \\
&= \sum_{n=1}^{\infty} \mu(E_n \setminus D_n) \\
&\leq \sum_{n=1}^{\infty} \mu(E_n),
\end{aligned}$$

where the second line above follows from the countable additivity of μ and the last line above follows from the first bullet point in 2.54. ∎

Note that countable subadditivity implies finite subadditivity: if μ is a measure on $(X, \mathcal{S})$ and $E_1, \ldots, E_M$ are sets in $\mathcal{S}$, then

$$\mu(E_1 \cup \cdots \cup E_M) \leq \mu(E_1) + \cdots + \mu(E_M),$$

as follows from applying the equation $\mu(\varnothing) = 0$ and countable subadditivity to the sequence $E_1, \ldots, E_M, \varnothing, \varnothing, \ldots$ of sets in $\mathcal{S}$.

The next result shows that measures behave well with increasing unions.

2.56 *Measure of an increasing union*

Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $E_1 \subset E_2 \subset \cdots$ is an increasing sequence of sets in $\mathcal{S}$. Then

$$\mu\Bigl(\bigcup_{n=1}^{\infty} E_n\Bigr) = \lim_{n \to \infty} \mu(E_n).$$

Proof If $\mu(E_n) = \infty$ for some $n \in \mathbf{Z}^+$, then the equation above holds because both sides equal ∞. Hence we can consider only the case where $\mu(E_n) < \infty$ for all $n \in \mathbf{Z}^+$.

For convenience, let $E_0 = \varnothing$. Then

$$\bigcup_{n=1}^{\infty} E_n = \bigcup_{m=1}^{\infty} (E_m \setminus E_{m-1}),$$

where the union on the right side is a disjoint union. Thus

$$\begin{aligned}
\mu\Bigl(\bigcup_{n=1}^{\infty} E_n\Bigr) &= \sum_{m=1}^{\infty} \mu(E_m \setminus E_{m-1}) \\
&= \lim_{n\to\infty} \sum_{m=1}^{n} \mu(E_m \setminus E_{m-1}) \\
&= \lim_{n\to\infty} \sum_{m=1}^{n} \bigl(\mu(E_m) - \mu(E_{m-1})\bigr) \\
&= \lim_{n\to\infty} \mu(E_n),
\end{aligned}$$

Another mew.

as desired. ■

Measures also behave well with respect to decreasing intersections (but see Exercise 10, which shows that why the hypothesis $\mu(E_1) < \infty$ is needed below).

2.57 *Measure of a decreasing intersection*

Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $E_1 \supset E_2 \supset \cdots$ is a decreasing sequence of sets in $\mathcal{S}$, with $\mu(E_1) < \infty$. Then

$$\mu\Bigl(\bigcap_{n=1}^{\infty} E_n\Bigr) = \lim_{n\to\infty} \mu(E_n).$$

Proof One of De Morgan's Laws (0.58) tells us that

$$E_1 \setminus \bigcap_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} (E_1 \setminus E_n).$$

Now $E_1 \setminus E_1 \subset E_1 \setminus E_2 \subset E_1 \setminus E_3 \subset \cdots$ is an increasing sequence of sets in $\mathcal{S}$. Thus 2.56, applied to the equation above, implies that

$$\mu\Bigl(E_1 \setminus \bigcap_{n=1}^{\infty} E_n\Bigr) = \lim_{n\to\infty} \mu(E_1 \setminus E_n).$$

Use the second bullet point of 2.54 to rewrite the equation above as

$$\mu(E_1) - \mu\Bigl(\bigcap_{n=1}^{\infty} E_n\Bigr) = \mu(E_1) - \lim_{n\to\infty} \mu(E_n),$$

which implies our desired result. ■

The next result is intuitively plausible—we expect that the measure of the union of two sets equals the measure of the first set plus the measure of the second set minus the measure of the set that has been counted twice.

2.58 *Measure of a union*

Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $D, E \in \mathcal{S}$, with $\mu(D \cap E) < \infty$. Then

$$\mu(D \cup E) = \mu(D) + \mu(E) - \mu(D \cap E).$$

Proof We have

$$D \cup E = \big(D \setminus (D \cap E)\big) \cup \big(E \setminus (D \cap E)\big) \cup (D \cap E).$$

The right side of the equation above is a disjoint union. Thus

$$\begin{aligned}\mu(D \cup E) &= \mu\big(D \setminus (D \cap E)\big) + \mu\big(E \setminus (D \cap E)\big) + \mu(D \cap E)\\ &= \big(\mu(D) - \mu(D \cap E)\big) + \big(\mu(E) - \mu(D \cap E)\big) + \mu(D \cap E)\\ &= \mu(D) + \mu(E) - \mu(D \cap E),\end{aligned}$$

as desired. ■

EXERCISES 2C

1 Explain why there does not exist a measure space $(X, \mathcal{S}, \mu)$ with the property that $\{\mu(E) : E \in \mathcal{S}\} = [0, 1)$.

Let $2^{\mathbf{Z}^+}$ denote the σ-algebra on $\mathbf{Z}^+$ consisting of all subsets of $\mathbf{Z}^+$.

2 Suppose μ is a measure on $(\mathbf{Z}^+, 2^{\mathbf{Z}^+})$. Prove that there is a sequence $w_1, w_2, \dots$ in $[0, \infty]$ such that

$$\mu(E) = \sum_{n \in E} w_n$$

for every set $E \subset \mathbf{Z}^+$.

3 Give an example of a measure μ on $(\mathbf{Z}^+, 2^{\mathbf{Z}^+})$ such that

$$\{\mu(E) : E \subset \mathbf{Z}^+\} = [0, 1].$$

4 Give an example of a measure space $(X, \mathcal{S}, \mu)$ such that

$$\{\mu(E) : E \in \mathcal{S}\} = \{\infty\} \cup \bigcup_{n=0}^{\infty} [3n, 3n+1].$$

5 Suppose $(X, \mathcal{S}, \mu)$ is a measure space such that $\mu(X) < \infty$. Prove that if $\mathcal{A}$ is a set of disjoint sets in $\mathcal{S}$ such that $\mu(A) > 0$ for every $A \in \mathcal{A}$, then $\mathcal{A}$ is a countable set.

6 Find all $c \in [3, \infty)$ such that there exists a measure space $(X, \mathcal{S}, \mu)$ with

$$\{\mu(E) : E \in \mathcal{S}\} = [0, 1] \cup [3, c].$$

7 Give an example of a measure space $(X, \mathcal{S}, \mu)$ such that

$$\{\mu(E) : E \in \mathcal{S}\} = [0, 1] \cup [3, \infty].$$

8 Give an example of a set X, a σ-algebra $\mathcal{S}$ of subsets of X, a set $\mathcal{A}$ of subsets of X such that the smallest σ-algebra on X containing $\mathcal{A}$ is $\mathcal{S}$, and two measures μ and ν on $(X, \mathcal{S})$ such that $\mu(A) = \nu(A)$ for all $A \in \mathcal{A}$ and $\mu(X) = \nu(X) < \infty$, but $\mu \neq \nu$.

9 Suppose μ and ν are measures on a measurable space $(X, \mathcal{S})$. Prove that $\mu + \nu$ is a measure on $(X, \mathcal{S})$. [Here $\mu + \nu$ is the usual sum of two functions: if $E \in \mathcal{S}$, then $(\mu + \nu)(E) = \mu(E) + \nu(E)$.]

10 Give an example of a measure space $(X, \mathcal{S}, \mu)$ and a decreasing sequence $E_1 \supset E_2 \supset \cdots$ of sets in $\mathcal{S}$ such that

$$\mu\Bigl(\bigcap_{n=1}^{\infty} E_n\Bigr) \neq \lim_{n \to \infty} \mu(E_n).$$

11 Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $C, D, E \in \mathcal{S}$ are such that

$$\mu(C \cap D) < \infty, \mu(C \cap E) < \infty, \text{ and } \mu(D \cap E) < \infty.$$

Find and prove a formula for $\mu(C \cup D \cup E)$ in terms of $\mu(C), \mu(D), \mu(E), \mu(C \cap D), \mu(C \cap E), \mu(D \cap E), \mu(C \cap D \cap E)$.

2D Lebesgue Measure

Additivity of outer measure on Borel sets

Recall that there exist disjoint sets $A, B \in \mathbf{R}$ such that $|A \cup B| \neq |A| + |B|$ (see 2.17). Thus outer measure, despite its name, is not a measure on the σ-algebra of all subsets of $\mathbf{R}$.

Our main goal in this section is to prove that outer measure, when restricted to the Borel subsets of $\mathbf{R}$, is a measure. Throughout this section, be careful about trying to simplify proofs by applying properties of measures to outer measure, even if those properties seem intuitively plausible. For example, there are subsets $A \subset B \subset \mathbf{R}$ with $|A| < \infty$ but $|B \setminus A| \neq |B| - |A|$ (compare to the second bullet point in 2.54).

The next result is our first step toward the goal of proving that outer measure restricted to the Borel sets is a measure.

2.59 ***Additivity of outer measure if one of the sets is open***

Suppose A and B are disjoint subsets of $\mathbf{R}$ and B is open. Then

$$|A \cup B| = |A| + |B|.$$

Proof We can assume that $|B| < \infty$ because otherwise both $|A \cup B|$ and $|A| + |B|$ equal ∞.

Subadditivity (see 2.8) implies that $|A \cup B| \leq |A| + |B|$. Thus we need to prove the inequality only in the other direction.

First consider the case where $B = (a, b)$ for some $a, b \in \mathbf{R}$ with $a < b$. We can assume that $a, b \notin A$ (because changing a set by at most two points does not change its outer measure). Let $I_1, I_2, \ldots$ be a sequence of open intervals whose union contains $A \cup B$. For each $n \in \mathbf{Z}^+$, let

$$J_n = I_n \cap (-\infty, a), \quad K_n = I_n \cap (a, b), \quad L_n = I_n \cap (b, \infty).$$

Then

$$\ell(I_n) = \ell(J_n) + \ell(K_n) + \ell(L_n).$$

Now $J_1, L_1, J_2, L_2, \ldots$ is a sequence of open intervals whose union contains A and $K_1, K_2, \ldots$ is a sequence of open intervals whose union contains B. Thus

$$\begin{aligned}\sum_{n=1}^{\infty} \ell(I_n) &= \sum_{n=1}^{\infty} \big(\ell(J_n) + \ell(L_n)\big) + \sum_{n=1}^{\infty} \ell(K_n) \\ &\geq |A| + |B|.\end{aligned}$$

The inequality above implies that $|A \cup B| \geq |A| + |B|$, completing the proof that $|A \cup B| = |A| + |B|$ in this special case.

Using induction on m, we can now conclude that if $m \in \mathbf{Z}^+$ and B is a union of m disjoint open intervals that are all disjoint from A, then $|A \cup B| = |A| + |B|$.

Now suppose B is an arbitrary open subset of $\mathbf{R}$ that is disjoint from A. Then $B = \bigcup_{n=1}^{\infty} I_n$ for some sequence of disjoint open intervals $I_1, I_2, \ldots$ (see 0.54), each of which is disjoint from A. Now for each $m \in \mathbf{Z}^+$ we have

$$\begin{aligned}|A \cup B| &\geq \Big|A \cup \Big(\bigcup_{n=1}^{m} I_n\Big)\Big| \\ &= |A| + \sum_{n=1}^{m} \ell(I_n).\end{aligned}$$

Thus

$$\begin{aligned}|A \cup B| &\geq |A| + \sum_{n=1}^{\infty} \ell(I_n) \\ &\geq |A| + |B|,\end{aligned}$$

completing the proof that $|A \cup B| = |A| + |B|$. ■

The next result shows that the outer measure of the disjoint union of two sets is what we expect if at least one of the two sets is closed.

2.60 *Additivity of outer measure if one of the sets is closed*

Suppose A and B are disjoint subsets of $\mathbf{R}$ and B is closed. Then

$$|A \cup B| = |A| + |B|.$$

Proof Suppose $I_1, I_2, \ldots$ is a sequence of open intervals whose union contains $A \cup B$. Let $D = \bigcup_{n=1}^{\infty} I_n$. Thus D is an open set with $A \cup B \subset D$. Hence $A \subset D \setminus B$, which implies that

2.61 $$|A| \leq |D \setminus B|.$$

Because $D \setminus B = D \cap (\mathbf{R} \setminus B)$, we know that $D \setminus B$ is an open set. Hence we can apply 2.59 to the disjoint union $D = B \cup (D \setminus B)$, getting

$$|D| = |B| + |D \setminus B|.$$

Adding $|B|$ to both sides of 2.61 and then using the equation above gives

$$\begin{aligned}|A| + |B| &\leq |D| \\ &\leq \sum_{n=1}^{\infty} \ell(I_n).\end{aligned}$$

Thus $|A| + |B| \leq |A \cup B|$, which implies that $|A| + |B| = |A \cup B|$. ■

Recall that the collection of Borel sets is the smallest σ-algebra on $\mathbf{R}$ that contains all open subsets of $\mathbf{R}$. The next result provides an extremely useful tool for approximating a Borel set by a closed set.

2.62 *Approximation of Borel sets from below by closed sets*

Suppose $B \subset \mathbf{R}$ is a Borel set. Then for every $\varepsilon > 0$, there exists a closed set $E \subset B$ such that $|B \setminus E| < \varepsilon$.

Proof Let

$$\mathcal{L} = \{D \subset \mathbf{R} : \text{for every } \varepsilon > 0, \text{ there exists a closed set } E \subset D \text{ such that } |D \setminus E| < \varepsilon.\}$$

The strategy of the proof is to show that $\mathcal{L}$ is a σ-algebra. Then because $\mathcal{L}$ contains every closed subset of $\mathbf{R}$ (if $D \subset \mathbf{R}$ is closed, take $E = D$ in the definition of $\mathcal{L}$), by taking complements we can conclude that $\mathcal{L}$ contains every open subset of $\mathbf{R}$ and thus every Borel subset of $\mathbf{R}$.

To get started with proving that $\mathcal{L}$ is a σ-algebra, we want to prove that $\mathcal{L}$ is closed under countable intersections. Thus suppose $D_1, D_2, \ldots$ is a sequence in $\mathcal{L}$. Let $\varepsilon > 0$. For each $n \in \mathbf{Z}^+$, there exists a closed set E_n such that

$$E_n \subset D_n \quad \text{and} \quad |D_n \setminus E_n| < \frac{\varepsilon}{2^n}.$$

Thus $\bigcap_{n=1}^\infty E_n$ is a closed set and

$$\bigcap_{n=1}^\infty E_n \subset \bigcap_{n=1}^\infty D_n \quad \text{and} \quad \Big(\bigcap_{n=1}^\infty D_n\Big) \setminus \Big(\bigcap_{n=1}^\infty E_n\Big) \subset \bigcup_{n=1}^\infty (D_n \setminus E_n).$$

The last set inclusion and the countable subadditivity of outer measure (see 2.8) imply that

$$\Big|\Big(\bigcap_{n=1}^\infty D_n\Big) \setminus \Big(\bigcap_{n=1}^\infty E_n\Big)\Big| < \varepsilon.$$

Thus $\bigcap_{n=1}^\infty D_n \in \mathcal{L}$, proving that $\mathcal{L}$ is closed under countable intersections.

Now we want to prove that $\mathcal{L}$ is closed under complementation. Suppose $D \in \mathcal{L}$ and $\varepsilon > 0$. We want to show that there is a closed subset of $\mathbf{R} \setminus D$ whose set difference with $\mathbf{R} \setminus D$ has outer measure less than ε, which will allow us to conclude that $\mathbf{R} \setminus D \in \mathcal{L}$.

First we consider the case where $|D| < \infty$. Let $E \subset D$ be a closed set such that $|D \setminus E| < \frac{\varepsilon}{2}$. The definition of outer measure implies that there exists an open set A such that $D \subset A$ and $|A| < |D| + \frac{\varepsilon}{2}$. Now $\mathbf{R} \setminus A$ is a closed set and $\mathbf{R} \setminus A \subset \mathbf{R} \setminus D$. Also, we have

$$\begin{aligned}(\mathbf{R} \setminus D) \setminus (\mathbf{R} \setminus A) &\subset A \setminus D \\ &\subset A \setminus E.\end{aligned}$$

Thus

$$\begin{aligned}|(\mathbf{R} \setminus D) \setminus (\mathbf{R} \setminus A)| &\leq |A \setminus E| \\ &= |A| - |E| \\ &= (|A| - |D|) + (|D| - |E|) \\ &< \frac{\varepsilon}{2} + |D \setminus E| \\ &< \varepsilon,\end{aligned}$$

where the equality in the second line above comes from applying 2.60 to the disjoint union $A = (A \setminus E) \cup E$, and the fourth line above uses subadditivity applied to the union $D = (D \setminus E) \cup E$. The last inequality above shows that $\mathbf{R} \setminus D \in \mathcal{L}$, as desired.

Now, still assuming that $D \in \mathcal{L}$ and $\varepsilon > 0$, we consider the case where $|D| = \infty$. For $n \in \mathbf{Z}^+$, let $D_n = D \cap [-n, n]$. Because $D_n \in \mathcal{L}$ and $|D_n| < \infty$, the previous case implies that $\mathbf{R} \setminus D_n \in \mathcal{L}$. Clearly $D = \bigcup_{n=1}^{\infty} D_n$. Thus

$$\mathbf{R} \setminus D = \bigcap_{n=1}^{\infty} (\mathbf{R} \setminus D_n).$$

Because $\mathcal{L}$ is closed under countable intersections, the equation above implies that $\mathbf{R} \setminus D \in \mathcal{L}$, which completes the proof that $\mathcal{L}$ is a σ-algebra. ∎

Now we can prove that the outer measure of the disjoint union of two sets is what we expect if at least one of the two sets is a Borel set.

2.63 ***Additivity of outer measure if one of the sets is a Borel set***

Suppose A and B are disjoint subsets of $\mathbf{R}$ and B is a Borel set. Then

$$|A \cup B| = |A| + |B|.$$

Proof Let $\varepsilon > 0$. Let E be a closed set such that $E \subset B$ and $|B \setminus E| < \varepsilon$ (see 2.62). Thus

$$\begin{aligned} |A \cup B| &\geq |A \cup E| \\ &= |A| + |E| \\ &= |A| + |B| - |B \setminus E| \\ &\geq |A| + |B| - \varepsilon, \end{aligned}$$

where the second and third lines above follow from 2.60 [use $B = (B \setminus E) \cup E$ for the third line].

Because the inequality above holds for all $\varepsilon > 0$, we have $|A \cup B| \geq |A| + |B|$, which implies that $|A \cup B| = |A| + |B|$. ∎

You have probably long suspected that not every subset of $\mathbf{R}$ is a Borel set. Now we can prove this suspicion.

2.64 ***Existence of a subset of*** $\mathbf{R}$ ***that is not a Borel set***

There exists a set $B \subset \mathbf{R}$ such that $|B| < \infty$ and B is not a Borel set.

Proof In the proof of 2.17, we showed that there exist sets $A, B \subset \mathbf{R}$ such that $|A \cup B| \neq |A| + |B|$. For any such sets, we must have $|B| < \infty$ because otherwise both $|A \cup B|$ and $|A| + |B|$ equal ∞ (as follows from the inequality $|B| \leq |A \cup B|$). Now 2.63 implies that B is not a Borel set. ∎

The tools we have constructed now allow us to prove that outer measure, when restricted to the Borel sets, is a measure.

2.65 *Outer measure is a measure on Borel sets*

Outer measure is a measure on $(\mathbf{R}, \mathcal{B})$, where $\mathcal{B}$ is the σ-algebra of Borel subsets of $\mathbf{R}$.

Proof Suppose $B_1, B_2, \dots$ is a disjoint sequence of Borel subsets of $\mathbf{R}$. Then for each $N \in \mathbf{Z}^+$ we have

$$\begin{aligned}\Bigl|\bigcup_{n=1}^{\infty} B_n\Bigr| &\geq \Bigl|\bigcup_{n=1}^{N} B_n\Bigr| \\ &= \sum_{n=1}^{N} |B_n|,\end{aligned}$$

where the first line above follows from 2.5 and the last line follows from 2.63 (and induction on N). Taking the limit as $N \to \infty$, we have $\bigl|\bigcup_{n=1}^{\infty} B_n\bigr| \geq \sum_{n=1}^{\infty} |B_n|$. The inequality in the other directions follows from countable subadditivity of outer measure (2.8). Hence

$$\Bigl|\bigcup_{n=1}^{\infty} B_n\Bigr| = \sum_{n=1}^{\infty} |B_n|.$$

Thus outer measure is a measure on the σ-algebra of Borel subsets of $\mathbf{R}$. ■

2.66 Definition *Lebesgue measure*

Lebesgue measure is the measure on $(\mathbf{R}, \mathcal{B})$, where $\mathcal{B}$ is the σ-algebra of Borel subsets of $\mathbf{R}$, that assigns to each Borel set its outer measure. The Lebesgue measure of a Borel set B is denoted $|B|$.

In other words, the Lebesgue measure of a set is the same as its outer measure, except that the term *Lebesgue measure* should not be applied to arbitrary sets but only to Borel sets (and also to what are called Lebesgue measurable sets, as we will soon see). Unlike outer measure, Lebesgue measure is actually a measure, as shown in 2.65. Lebesgue measure is named in honor of its inventor, Henri Lebesgue.

The cathedral in Beauvais, France, where Henri Lebesgue (1875–1941) was born. Much of what we call Lebesgue measure and Lebesgue integration was developed by Lebesgue in his 1902 PhD thesis. Émile Borel was Lebesgue's PhD thesis advisor.

Lebesgue Measurable Sets

We have accomplished the major goal of this section, which was to show that outer measure restricted to Borel sets is a measure. As we will see in this subsection, outer measure is actually a measure on a somewhat larger class of sets called the Lebesgue measurable sets.

Many different definitions of Lebesgue measure exist in the mathematics literature. These definitions are all equivalent—the definition of Lebesgue measure in one approach becomes a theorem in another approach to this topic. The approach chosen here has the advantage of emphasizing that a Lebesgue measurable set differs from a Borel set by a set with outer measure 0. The attitude here is that sets with outer measure 0 should be considered small sets that do not matter much.

2.67 Definition ***Lebesgue measurable set***

A set $A \subset \mathbf{R}$ is called *Lebesgue measurable* if there exists a Borel set $B \subset A$ such that $|A \setminus B| = 0$.

Every Borel set is Lebesgue measurable because if $A \subset \mathbf{R}$ is a Borel set, then we can take $B = A$ in the definition above.

The result below gives several equivalent conditions for being Lebesgue measurable. The equivalence of (a) and (d) is just our definition and thus will not be discussed in the proof.

Although there exist Lebesgue measurable sets that are not Borel sets, you are unlikely to encounter one. The most important application of the result below is that if $A \subset \mathbf{R}$ is a Borel set, then A satisfies conditions (b), (c), (e), and (f). Condition (c) implies that every Borel set is almost a countable union of closed sets, and condition (f) implies that every Borel set is almost a countable intersection of open sets.

2.68 ***Equivalences for being a Lebesgue measurable set***

Suppose $A \subset \mathbf{R}$. Then the following are equivalent:

(a) A is Lebesgue measurable.

(b) For each $\varepsilon > 0$, there exists a closed set $B \subset A$ with $|A \setminus B| < \varepsilon$.

(c) There exist closed sets $B_1, B_2 \ldots$ contained in A such that $\left|A \setminus \bigcup_{n=1}^{\infty} B_n\right| = 0$.

(d) There exists a Borel set $B \subset A$ such that $|A \setminus B| = 0$.

(e) For each $\varepsilon > 0$, there exists an open set $B \supset A$ such that $|B \setminus A| < \varepsilon$.

(f) There exist open sets $B_1, B_2 \ldots$ containing A such that $\left|\left(\bigcap_{n=1}^{\infty} B_n\right) \setminus A\right| = 0$.

(g) There exists a Borel set $B \supset A$ such that $|B \setminus A| = 0$.

Proof Let $\mathcal{L}$ denote the collection of sets $A \subset \mathbf{R}$ that satisfy (b). We have already proved that every Borel set is in $\mathcal{L}$—see 2.62. As a key part of that proof, which we will freely use in this proof, we showed that $\mathcal{L}$ is a σ-algebra on $\mathbf{R}$ (see the proof of 2.62). In addition to containing the Borel sets, $\mathcal{L}$ contains every set with outer measure 0 [because if $|A| = 0$, we can take $B = \varnothing$ in (b)].

(b) $\Longrightarrow$ (c): Suppose (b) holds. Thus for each $k \in \mathbf{Z}^+$, there exists a closed set $B_k \subset A$ such that $|A \setminus B_k| < \frac{1}{k}$. Now

$$A \setminus \bigcup_{n=1}^{\infty} B_n \subset A \setminus B_k$$

for each $k \in \mathbf{Z}^+$. Thus $|A \setminus \bigcup_{n=1}^{\infty} B_n| \leq |A \setminus B_k| \leq \frac{1}{k}$ for each $k \in \mathbf{Z}^+$. Hence $|A \setminus \bigcup_{n=1}^{\infty} B_n| = 0$, completing the proof that (b) implies (c).

(c) $\Longrightarrow$ (d): Because every countable union of closed sets is a Borel set, we see that (c) implies (d).

(d) $\Longrightarrow$ (b): Suppose (d) holds. Thus there exists a Borel set $B \subset A$ such that $|A \setminus B| = 0$. Now

$$A = B \cup (A \setminus B).$$

We know that $B \in \mathcal{L}$ (because B is a Borel set) and $A \setminus B \in \mathcal{L}$ (because $A \setminus B$ has outer measure 0). Because $\mathcal{L}$ is a σ-algebra, the displayed equation above implies that $A \in \mathcal{L}$. In other words, (b) holds, completing the proof that (d) implies (b).

At this stage of the proof, we now know that (b) $\Longleftrightarrow$ (c) $\Longleftrightarrow$ (d).

(b) $\Longrightarrow$ (e): Suppose (b) holds. Thus $A \in \mathcal{L}$. Let $\varepsilon > 0$. Then because $\mathbf{R} \setminus A \in \mathcal{L}$ (which holds because $\mathcal{L}$ is closed under complementation), there exists a closed set $E \subset \mathbf{R} \setminus A$ such that

$$|(\mathbf{R} \setminus A) \setminus E| < \varepsilon.$$

Thus $\mathbf{R} \setminus E$ is an open set with $\mathbf{R} \setminus E \supset A$. Because $(\mathbf{R} \setminus E) \setminus A \subset (\mathbf{R} \setminus A) \setminus E$, the inequality above implies that $|(\mathbf{R} \setminus E) \setminus A| < \varepsilon$. Thus (e) holds, completing the proof that (b) implies (e).

(e) $\Longrightarrow$ (f): Suppose (e) holds. Thus for each $k \in \mathbf{Z}^+$, there exists an open set $B_k \supset A$ such that $|B_k \setminus A| < \frac{1}{k}$. Now

$$\Bigl(\bigcap_{n=1}^{\infty} B_n\Bigr) \setminus A \subset B_k \setminus A$$

for each $k \in \mathbf{Z}^+$. Thus $\bigl|\bigl(\bigcap_{n=1}^{\infty} B_n\bigr) \setminus A\bigr| \leq |B_k \setminus A| \leq \frac{1}{k}$ for each $k \in \mathbf{Z}^+$. Hence $\bigl|\bigl(\bigcap_{n=1}^{\infty} B_n\bigr) \setminus A\bigr| = 0$, completing the proof that (e) implies (f).

(f) $\Longrightarrow$ (g): Because every countable intersection of open sets is a Borel set, we see that (f) implies (g).

(g) $\Longrightarrow$ (b): Suppose (g) holds. Thus there exists a Borel set $B \supset A$ such that $|B \setminus A| = 0$. Now

$$A = B \cap \bigl(\mathbf{R} \setminus (B \setminus A)\bigr).$$

We know that $B \in \mathcal{L}$ (because B is a Borel set) and $B \setminus A \in \mathcal{L}$ (because this set is the complement of a set with outer measure 0). Because $\mathcal{L}$ is a σ-algebra, the displayed equation above implies that $A \in \mathcal{L}$. In other words, (b) holds, completing the proof that (g) implies (b).

Our chain of implications now shows that (b) through (g) are all equivalent. ∎

In practice, the most useful part of Exercise 14 is the result that every Borel set with finite measure is almost a finite disjoint union of bounded open intervals.

In addition to the equivalences in the previous result, see Exercise 13 in this section for another condition that is equivalent to being Lebesgue measurable. Also see Exercise 14, which shows that a set with finite outer measure is Lebesgue measurable if and only if it is almost a finite disjoint union of bounded open intervals.

Now we can show that outer measure is a measure on the Lebesgue measurable sets.

2.69 *Outer measure is a measure on Lebesgue measurable sets*

- The set $\mathcal{L}$ of Lebesgue measurable subsets of $\mathbf{R}$ is a σ-algebra on $\mathbf{R}$.
- Outer measure is a measure on $(\mathbf{R}, \mathcal{L})$.

Proof Because (a) and (b) are equivalent in 2.68, the set $\mathcal{L}$ of Lebesgue measurable subsets of $\mathbf{R}$ equals the collection of sets satisfying (b) in 2.68. As noted in the first paragraph of the proof of 2.68, this set is a σ-algebra on $\mathbf{R}$.

To prove the second bullet point, suppose $A_1, A_2, \ldots$ is a disjoint sequence of Lebesgue measurable sets. By the definition of Lebesgue measurable set (2.67), for each $n \in \mathbf{Z}^+$ there exists a Borel set $B_n \subset A_n$ such that $|A_n \setminus B_n| = 0$. Now

$$\begin{aligned}\Bigl|\bigcup_{n=1}^{\infty} A_n\Bigr| &\geq \Bigl|\bigcup_{n=1}^{\infty} B_n\Bigr| \\ &= \sum_{n=1}^{\infty} |B_n| \\ &= \sum_{n=1}^{\infty} |A_n|,\end{aligned}$$

where the second line above holds because $B_1, B_2, \ldots$ is a disjoint sequence of Borel sets and outer measure is a measure on the Borel sets (see 2.65); the last line above holds because $B_n \subset A_n$ and by subadditivity of outer measure (see 2.8) we have $|A_n| = |B_n \cup (A_n \setminus B_n)| \leq |B_n| + |A_n \setminus B_n| = |B_n|$.

The inequality above, combined with countable subadditivity of outer measure (see 2.8), implies that $\bigl|\bigcup_{n=1}^{\infty} A_n\bigr| = \sum_{n=1}^{\infty} |A_n|$, completing the proof that outer measure is a measure on the Lebesgue measurable sets. ■

If A is a set with outer measure 0, then A is Lebesgue measurable (because we can take $B = \emptyset$ in the definition 2.67). Our definition of the Lebesgue measurable sets thus implies that the set of Lebesgue measurable sets is the smallest σ-algebra on $\mathbf{R}$ containing the Borel sets and the sets with outer measure 0. Thus the set of Lebesgue measurable sets is also the smallest σ-algebra on $\mathbf{R}$ containing the open sets and the sets with outer measure 0.

Because outer measure is not even finitely additive (see 2.17), 2.69 implies that there exist subsets of $\mathbf{R}$ that are not Lebesgue measurable.

We previously defined Lebesgue measure as outer measure restricted to the Borel sets (see 2.66). The term *Lebesgue measure* is sometimes used in mathematical literature with the meaning as we previously defined it and is sometimes used with the following meaning.

2.70 Definition ***Lebesgue measure***

Lebesgue measure is the measure on $(\mathbf{R}, \mathcal{L})$, where $\mathcal{L}$ is the σ-algebra of Lebesgue measurable subsets of $\mathbf{R}$, that assigns to each Lebesgue measurable set its outer measure. The Lebesgue measure of a Lebesgue measurable set B is denoted $|B|$.

The two definitions of *Lebesgue measure* disagree only on the domain of the measure—is the σ-algebra the Borel sets or the Lebesgue measurable sets? You may be able to tell which is intended from the context. In this book, the domain will be clearly specified (usually the Borel sets).

If you are reading a mathematics paper and the domain for Lebesgue measure is not specified, then it probably does not matter whether you use the Borel sets or the Lebesgue measurable sets (because every Lebesgue measurable set differs from a Borel set by a set with outer measure 0, and when dealing with measures, what happens on a set with measure 0 usually does not matter). Because all sets that arise from the usual operations of analysis are Borel sets, you may want to assume that *Lebesgue measure* means outer measure on the Borel sets, unless what you are reading explicitly states otherwise.

A mathematics paper may also refer to a *measurable* subset of $\mathbf{R}$, without further explanation. Unless some other σ-algebra is clear from the context, the author probably means the Borel sets or the Lebesgue measurable sets. Again, the choice probably will not matter, but using the Borel sets can be cleaner and simpler.

The emphasis in some textbooks on Lebesgue measurable sets instead of Borel sets probably stems from the historical development of the subject, rather than from any serious use of Lebesgue measurable sets that are not Borel sets.

Lebesgue measure on the Lebesgue measurable sets does have one small advantage over Lebesgue measure on the Borel sets: Every subset of a set with (outer) measure 0 is Lebesgue measurable but is not necessarily a Borel set. However, any natural process that produces a subset of $\mathbf{R}$ will produce a Borel set. Thus this small advantage does not come up in practice.

Cantor Set

Every countable set has outer measure 0 (see 2.4). A reasonable question arises about whether the converse holds. In other words, is every set with outer measure 0 countable? The Cantor set, which is introduced in this subsection, provides the answer to this question.

The Cantor set also gives counterexamples to other reasonable conjectures. For example, Exercise 18 in this section shows that the sum of two sets with Lebesgue measure 0 can have positive Lebesgue measure.

2.71 Definition ***Cantor set***

The *Cantor set* is $[0,1] \setminus \left(\bigcup_{n=1}^{\infty} T_n\right)$, where $T_1 = \left(\frac{1}{3}, \frac{2}{3}\right)$ and T_n for $n > 1$ is the union of the middle-third open intervals in the intervals of $[0,1] \setminus \left(\bigcup_{j=1}^{n-1} T_j\right)$.

One way to envision the Cantor set is to start with the interval $[0,1]$ and then consider the process that removes at each step the middle-third open intervals of all intervals left from the previous step. At the first step, we remove $T_1 = \left(\frac{1}{3}, \frac{2}{3}\right)$.

T_1 is shown in red.

After that first step, we have $[0,1] \setminus T_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. Thus we take the middle-third open intervals of $[0, \frac{1}{3}]$ and $[\frac{2}{3}, 1]$. In other words, we have

$$T_2 = \left(\tfrac{1}{9}, \tfrac{2}{9}\right) \cup \left(\tfrac{7}{9}, \tfrac{8}{9}\right).$$

$T_1 \cup T_2$ is shown in red.

Now $[0,1] \setminus (T_1 \cup T_2) = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$. Thus

$$T_3 = \left(\tfrac{1}{27}, \tfrac{2}{27}\right) \cup \left(\tfrac{7}{27}, \tfrac{8}{27}\right) \cup \left(\tfrac{19}{27}, \tfrac{20}{27}\right) \cup \left(\tfrac{25}{27}, \tfrac{26}{27}\right).$$

$T_1 \cup T_2 \cup T_3$ is shown in red.

Base 3 representations provide a useful way to think about the Cantor set. Just as $\frac{1}{10} = 0.1 = 0.09999\ldots$ in the decimal representation, base 3 representations are not unique for fractions whose denominator is a power of 3. For example, $\frac{1}{3} = 0.1_3 = 0.02222\ldots_3$, where the subscript 3 denotes a base 3 representations.

Notice that T_1 equals the set of numbers in $[0,1]$ whose base 3 representations have 1 in the first digit after the decimal point (for those numbers that have two base 3 representations, this mean that both such representations must have 1 in the first digit). Also, $T_1 \cup T_2$ equals the set of numbers in $[0,1]$ whose base 3 representations have 1 in the first digit or the second digit after the decimal point. And so on. Hence $\bigcup_{n=1}^{\infty} T_n$ is the set of numbers in $[0,1]$ whose base 3 representations have a 1 somewhere.

Thus we have the following description of the Cantor set. In the following result, the phrase *a base* 3 *representation* indicates that if a number has two base 3 representations, then it is in the Cantor set if and only if at least one of them contains no 1s. For example, both $\frac{1}{3}$ (which equals $0.02222\ldots_3$) and $\frac{2}{3}$ (which equals 0.2_3) are in the Cantor set.

2.72 *Base 3 description of the Cantor set*

The Cantor set equals the set of numbers in $[0,1]$ that have a base 3 representation containing only 0s and 2s.

It is unknown whether or not every number in the Cantor set is either rational or transcendental (meaning not the root of a polynomial with integer coefficients).

The two endpoints of each interval in each T_n is in the Cantor set. However, many elements of the Cantor set are not endpoints of any interval in any T_n. For example, Exercise 15 asks you to show that $\frac{1}{4}$ and $\frac{9}{13}$ are in the Cantor set; neither of those numbers is an endpoint of any interval in any T_n. An example of an irrational number in the Cantor set is $\displaystyle\sum_{n=1}^{\infty} \frac{2}{3^{n!}}$.

2.73 *Properties of the Cantor set*

- The Cantor set is a closed subset of $\mathbf{R}$.
- The Cantor set has Lebesgue measure 0.
- The Cantor set is uncountable.
- The Cantor set contains no interval with more than one element.
- Every open interval of $\mathbf{R}$ contains either infinitely many or no elements in the Cantor set.

Proof Each set T_n used in the definition of the Cantor set is a union of open intervals. Thus each T_n is open. Thus $\bigcup_{n=1}^{\infty} T_n$ is open, and hence its complement is closed. The Cantor set equals $[0,1] \cap \big(\mathbf{R} \setminus \bigcup_{n=1}^{\infty} T_n\big)$, which is the intersection of two closed sets. Thus the Cantor set is closed, completing the proof of the first bullet point.

By induction on n, each T_n is the union of 2^{n-1} disjoint open intervals, each of which has length $\frac{1}{3^n}$. Thus $|T_n| = \frac{2^{n-1}}{3^n}$. The sets $T_1, T_2, \dots$ are disjoint. Hence

$$\begin{aligned}
\Big|\bigcup_{n=1}^{\infty} T_n\Big| &= \frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \cdots \\
&= \frac{1}{3}\Big(1 + \frac{2}{3} + \frac{4}{9} + \cdots\Big) \\
&= \frac{1}{3} \cdot \frac{1}{1 - \frac{2}{3}} \\
&= 1.
\end{aligned}$$

Thus the Cantor set, which equals $[0,1] \setminus \bigcup_{n=1}^{\infty} T_n$, has Lebesgue measure $1 - 1$ (by 2.54). In other words, the Cantor set has Lebesgue measure 0, completing the proof of the second bullet point.

Each number in the Cantor set has a unique base 3 representation containing only 0s and 2s (by 2.72; for those numbers that have two base 3 representations, one of them must contain a 1). In that base 3 representation containing only 0s and 2s, replace each 2 by 1 and consider the resulting string of digits as representing a base 2 number. This gives a mapping of the Cantor set onto $[0,1]$. Because $[0,1]$ is uncountable (see 2.16), this implies that the Cantor set is uncountable, thus proving the third bullet point.

A set with Lebesgue measure 0 cannot contain an interval that has more than one element. Thus the second bullet point implies the fourth bullet point.

The proof of the last bullet point is left as an exercise for the reader. ■

EXERCISES 2D

1 (a) Show that the set consisting of those numbers in $(0,1)$ that have a decimal expansion containing one hundred consecutive 4s is a Borel subset of **R**.

(b) What is the Lebesgue measure of the set in part (a)?

2 Prove that there exists a bounded set $A \subset \mathbf{R}$ such that

$$|B| \leq |A| - 1$$

for every closed set $B \subset A$.

3 Prove that there exists a set $A \subset \mathbf{R}$ such that

$$|B \setminus A| = \infty$$

for every open set B that contains A.

4 The phrase *nontrivial interval* is used to denote an interval of **R** that contains more than one element. Recall that an interval might be open, closed, or neither.

(a) Prove that the union of each collection of nontrival intervals of **R** is equal to the union of a countable subset of that collection.

(b) Prove that the union of each collection of nontrivial intervals of **R** is a Borel set.

(c) Prove that there exists a collection of closed intervals of **R** whose union is not a Borel set.

5 Prove that if $B \subset \mathbf{R}$ is Lebesgue measurable, then

$$|B| = \sup\{|A| : A \text{ is a closed bounded subset of } B\}.$$

6 Prove that if $A \subset \mathbf{R}$ is Lebesgue measurable, then there exists an increasing sequence $B_1 \subset B_2 \subset \cdots$ of closed sets contained in A such that

$$\left| A \setminus \bigcup_{n=1}^{\infty} B_n \right| = 0.$$

7 Prove that if $A \subset \mathbf{R}$ is Lebesgue measurable, then there exists a decreasing sequence $B_1 \supset B_2 \supset \cdots$ of open sets containing A such that

$$\Big|\Big(\bigcap_{n=1}^{\infty} B_n\Big) \setminus A\Big| = 0.$$

8 Prove that the collection of Lebesgue measurable subsets of $\mathbf{R}$ is translation invariant. More precisely, prove that if $A \subset \mathbf{R}$ is Lebesgue measurable and $t \in \mathbf{R}$, then $t + A$ is Lebesgue measurable.

9 Prove that the collection of Lebesgue measurable subsets of $\mathbf{R}$ is dilation invariant. More precisely, prove that if $A \subset \mathbf{R}$ is Lebesgue measurable and $t \in \mathbf{R}$, then tA (which is defined to be $\{ta : a \in A\}$) is Lebesgue measurable.

10 Prove that if A and B are disjoint subsets of $\mathbf{R}$ and B is Lebesgue measurable, then $|A \cup B| = |A| + |B|$.

11 Prove that if $A \subset \mathbf{R}$ and $|A| > 0$, then there exists a subset of A that is not Lebesgue measurable.

12 Suppose $b < c$ and $A \subset [b, c]$. Prove that A is Lebesgue measurable if and only if

$$|A| + |[b, c] \setminus A| = c - b.$$

13 Suppose $A \subset \mathbf{R}$. Prove that A is Lebesgue measurable if and only if

$$|[-n, n] \cap A| + |[-n, n] \setminus A| = 2n$$

for every $n \in \mathbf{Z}^+$.

14 Suppose $A \subset \mathbf{R}$ and $|A| < \infty$. Prove that A is Lebesgue measurable if and only if for every $\varepsilon > 0$ there exists a set B that is the union of finitely many disjoint bounded open intervals such that $|A \setminus B| + |B \setminus A| < \varepsilon$.

15 Show that $\frac{1}{4}$ and $\frac{9}{13}$ are both in the Cantor set.

16 Show that $\frac{13}{17}$ is not in the Cantor set.

17 List the eight open intervals whose union equals T_4 in the definition of the Cantor set (2.71).

18 Let C denote the Cantor set. Prove that $\frac{1}{2}C + \frac{1}{2}C = [0, 1]$.

19 Prove the last bullet point in 2.73.

20 Prove or give a counterexample: There exists a function $f\colon \mathbf{R} \to \mathbf{R}$ such that the image under f of every nonempty open interval equals $\mathbf{R}$.

2E Functions on Measure Spaces

Recall that a measurable space is a pair $(X, \mathcal{S})$, where X is a set and $\mathcal{S}$ is a σ-algebra on X. We defined a function $f\colon X \to \mathbf{R}$ to be $\mathcal{S}$-measurable if $f^{-1}(B) \in \mathcal{S}$ for every Borel set $B \subset \mathbf{R}$. In Section 2B we proved some results about $\mathcal{S}$-measurable functions; this was before we had introduced the notion of a measure.

In this section, we return to study measurable functions, but now with an emphasis on results that depend upon measures. The highlights of this section are the proofs of Egorov's Theorem and Luzin's Theorem.

Pointwise and Uniform Convergence

We begin this section with some definitions that you probably saw in an earlier course.

2.74 Definition ***pointwise convergence; uniform convergence***

Suppose X is a set, $f_1, f_2, \dots$ is a sequence of functions from X to $\mathbf{R}$, and f is a function from X to $\mathbf{R}$.

- The sequence $f_1, f_2, \dots$ *converges pointwise* on X to f if

  $$\lim_{n\to\infty} f_n(x) = f(x)$$

  for each $x \in X$.

  In other words, $f_1, f_2, \dots$ *converges pointwise* on X to f if for each $x \in X$ and every $\varepsilon > 0$, there exists $N \in \mathbf{Z}^+$ such that $|f_n(x) - f(x)| < \varepsilon$ for all integers $n \geq N$.

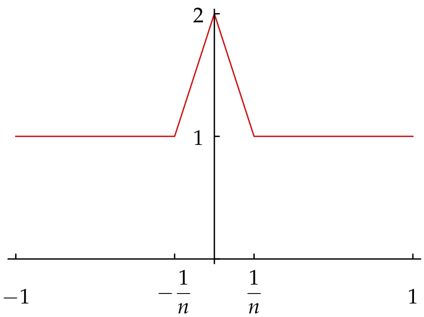
- The sequence $f_1, f_2, \dots$ *converges uniformly* on X to f if for every $\varepsilon > 0$, there exists $N \in \mathbf{Z}^+$ such that $|f_n(x) - f(x)| < \varepsilon$ for all integers $n \geq N$ and all $x \in X$.

2.75 Example *a sequence converging pointwise but not uniformly*

Suppose $f_n\colon [-1, 1] \to \mathbf{R}$ is the function whose graph is shown here and $f\colon [-1, 1] \to \mathbf{R}$ is the function defined by

$$f(x) = \begin{cases} 1 & \text{if } x \neq 0, \\ 2 & \text{if } x = 0. \end{cases}$$

Then $f_1, f_2, \dots$ converges pointwise on $[-1, 1]$ to f but $f_1, f_2, \dots$ does not converge uniformly on $[-1, 1]$ to f, as you should verify.



The graph of f_n.

Like the difference between continuity and uniform continuity, the difference between pointwise convergence and uniform convergence lies in the order of the quantifiers. Take a moment to examine the definitions carefully. If a sequence of functions converges uniformly on some set, then it also converges pointwise on the same set; however, the converse is not true, as shown by Example 2.75.

Example 2.75 also shows that the pointwise limit of continuous functions need not be continuous. However, the next result tells us that the uniform limit of continuous functions is continuous.

2.76 *Uniform limit of continuous functions is continuous*

Suppose $B \subset \mathbf{R}$ and $f_1, f_2, \dots$ is a sequence of functions from B to $\mathbf{R}$ that converges uniformly on B to a function $f\colon B \to \mathbf{R}$. Suppose $b \in B$ and f_n is continuous at b for each $n \in \mathbf{Z}^+$. Then f is continuous at b.

Proof Suppose $\varepsilon > 0$. Let $N \in \mathbf{Z}^+$ be such that $|f_N(x) - f(x)| < \frac{\varepsilon}{3}$ for all $x \in B$. Because f_N is continuous at b, there exists $\delta > 0$ such that $|f_N(x) - f_N(b)| < \frac{\varepsilon}{3}$ for $x \in (b - \delta, b + \delta) \cap B$.

Now suppose $x \in (b - \delta, b + \delta) \cap B$. Then

$$\begin{aligned} |f(x) - f(b)| &\leq |f(x) - f_N(x)| + |f_N(x) - f_N(b)| + |f_N(b) - f(b)| \\ &< \varepsilon. \end{aligned}$$

Thus f is continuous at b. ■

Egorov's Theorem

Russian mathematician Dmitri Egorov (1869–1931) proved the theorem below in 1911. You may encounter some books that spell his last name as Egoroff.

A sequence of functions that converges pointwise need not converge uniformly. However, the next result says that a pointwise convergent sequence of functions on a measure space almost converges uniformly, in the sense that it converges uniformly except on a set that can have arbitrarily small measure.

As an example of the next result, consider Lebesgue measure λ on the interval $[-1, 1]$ and the sequence of functions $f_1, f_2, \dots$ in Example 2.75 that converges pointwise but not uniformly on $[-1, 1]$. Suppose $\varepsilon > 0$. Then taking $E = [-1, -\frac{\varepsilon}{4}] \cup [\frac{\varepsilon}{4}, 1]$, we have $\lambda([-1, 1] \setminus E) < \varepsilon$ and $f_1, f_2, \dots$ converges uniformly on E, as in the conclusion of the next result.

2.77 *Egorov's Theorem*

Suppose $(X, \mathcal{S}, \mu)$ is a measure space with $\mu(X) < \infty$. Suppose $f_1, f_2, \dots$ is a sequence of $\mathcal{S}$-measurable functions from X to $\mathbf{R}$ that converges pointwise on X to a function $f\colon X \to \mathbf{R}$. Then for every $\varepsilon > 0$, there exists a set $E \in \mathcal{S}$ such that $\mu(X \setminus E) < \varepsilon$ and $f_1, f_2, \dots$ converges uniformly to f on E.

Proof Suppose $\varepsilon > 0$. Temporarily fix $k \in \mathbf{Z}^+$. The definition of pointwise convergence implies that

2.78 $$\bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \{x \in X : |f_n(x) - f(x)| < \tfrac{1}{k}\} = X.$$

For $N \in \mathbf{Z}^+$, let

$$A_{N,k} = \bigcap_{n=N}^{\infty} \{x \in X : |f_n(x) - f(x)| < \tfrac{1}{k}\}.$$

Then clearly $A_{1,k} \subset A_{2,k} \subset \cdots$ is an increasing sequence of sets and 2.78 can be rewritten as

$$\bigcup_{N=1}^{\infty} A_{N,k} = X.$$

The equation above implies (by 2.56) that $\lim_{N\to\infty} \mu(A_{N,k}) = \mu(X)$. Thus there exists $N_k \in \mathbf{Z}^+$ such that

$$\mu(X) - \mu(A_{N_k,k}) < \frac{\varepsilon}{2^k}.$$

Now let

$$E = \bigcap_{k=1}^{\infty} A_{N_k,k}.$$

Then

$$\begin{aligned} \mu(X \setminus E) &= \mu\Bigl(X \setminus \bigcap_{k=1}^{\infty} A_{N_k,k}\Bigr) \\ &= \mu\Bigl(\bigcup_{k=1}^{\infty} (X \setminus A_{N_k,k})\Bigr) \\ &\leq \sum_{k=1}^{\infty} \mu(X \setminus A_{N_k,k}) \\ &< \varepsilon. \end{aligned}$$

To complete the proof, we must verify that $f_1, f_2, \ldots$ converges uniformly to f on E. To do this, suppose $\varepsilon' > 0$. Let $k \in \mathbf{Z}^+$ be such that $\frac{1}{k} < \varepsilon'$. Then $E \subset A_{N_k,k}$, which implies that

$$|f_n(x) - f(x)| < \frac{1}{k} < \varepsilon'$$

for all $n \geq N_k$ and all $x \in E$. Hence $f_1, f_2, \ldots$ does indeed converge uniformly to f on E. ∎

Approximation by Simple Functions

2.79 Definition *simple function*

A function is called *simple* if it takes on only finitely many values.

Suppose $(X, \mathcal{S})$ is a measurable space, $f\colon X \to \mathbf{R}$ is a simple function, and $c_1, \dots, c_n$ are the distinct nonzero values of f. Then

$$f = c_1 \chi_{E_1} + \cdots + c_n \chi_{E_n},$$

where $E_k = f^{-1}(\{c_k\})$. Thus f is an $\mathcal{S}$-measurable function if and only if $E_1, \dots, E_n \in \mathcal{S}$ (as you should verify).

2.80 ***Approximation by simple functions***

Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $f\colon X \to [-\infty, \infty]$ is $\mathcal{S}$-measurable. Then there exists a sequence $f_1, f_2, \dots$ of functions from X to $\mathbf{R}$ such that

- each f_n is a simple $\mathcal{S}$-measurable function;
- $|f_n(x)| \leq |f_{n+1}(x)| \leq |f(x)|$ for all $n \in \mathbf{Z}^+$ and all $x \in X$;
- $\lim\limits_{n \to \infty} f_n(x) = f(x)$ for every $x \in X$;
- $f_1, f_2, \dots$ converges uniformly on X to f if f is bounded.

Proof The idea of the proof is that for each $n \in \mathbf{Z}^+$ and $k \in \mathbf{Z}$, the interval $[k, k+1)$ is divided into 2^n equally-sized half-open subintervals. If $f(x) \in [0, n]$, we define $f_n(x)$ to be the left endpoint of the subinterval into which $f(x)$ falls; if $f(x) \in [-n, 0)$, we define $f_n(x)$ to be the right endpoint of the subinterval into which $f(x)$ falls; and if $|f(x)| > n$, we define $f_n(x)$ to be 0. Specifically, let

$$f_n(x) = \begin{cases} \dfrac{m}{2^n} & \text{if } 0 \leq f(x) \leq n \text{ and } m \in \mathbf{Z} \text{ is such that } f(x) \in \Big[\dfrac{m}{2^n}, \dfrac{m+1}{2^n}\Big), \\ \dfrac{m+1}{2^n} & \text{if } -n \leq f(x) < 0 \text{ and } m \in \mathbf{Z} \text{ is such that } f(x) \in \Big[\dfrac{m}{2^n}, \dfrac{m+1}{2^n}\Big), \\ n & \text{if } f(x) > n, \\ -n & \text{if } f(x) < -n. \end{cases}$$

Each $f^{-1}\big([\frac{m}{2^n}, \frac{m+1}{2^n})\big) \in \mathcal{S}$ because f is $\mathcal{S}$-measurable. Thus each f_n is an $\mathcal{S}$-measurable simple function.

The second bullet point is satisfied because of how we have defined f_n.

The definition of f_n implies that

$$|f_n(x) - f(x)| \leq \frac{1}{2^n} \quad \text{for all } x \in X \text{ such that } f(x) \in [-n, n].$$

Thus we see that $\lim_{n \to \infty} f_n(x) = f(x)$ for every $x \in X$. Furthermore, the inequality above shows that $f_1, f_2, \dots$ converges uniformly on X to f if f is bounded. ■

Luzin's Theorem

Russian mathematician Nikolai Luzin (1883–1950) proved the theorem below in 1912. Most mathematics literature in English refers to the result below as Lusin's Theorem. *However,* Luzin *is the correct transliteration from Russian into English;* Lusin *is the transliteration into German.*

Our next result is surprising. It says that an arbitrary Borel measurable function is almost continuous, in the sense that its restriction to a large closed set is continuous. Here, the phrase *large closed set* means that we can take the complement of the closed set to have arbitrarily small measure.

Be careful about the interpretation of the conclusion of Luzin's Theorem that $f|_B$ is a continuous function on B. This is not the same as saying that f (on its original domain) is continuous at each point of B. For example, $\chi_{\mathbf{Q}}$ is discontinuous at every point of $\mathbf{R}$. However, $\chi_{\mathbf{Q}}|_{\mathbf{R}\setminus\mathbf{Q}}$ is a continuous function on $\mathbf{R} \setminus \mathbf{Q}$ (because this function is identically 0 on its domain).

2.81 *Luzin's Theorem*

Suppose $A \subset \mathbf{R}$ and $f\colon A \to \mathbf{R}$ is a Borel measurable function. Then for every $\varepsilon > 0$, there exists a closed set $B \subset A$ such that $|A \setminus B| < \varepsilon$ and $f|_B$ is a continuous function on B.

Proof Because Borel sets can be approximated from below by closed sets, it will be enough to prove the result with *Borel set* replacing *closed set*. Specifically, suppose $\varepsilon > 0$ and $B \subset A$ is a Borel set such that $|A \setminus B| < \frac{\varepsilon}{2}$ and $f|_B$ is continuous. Then there exists a closed set $B' \subset B$ such that $|B \setminus B'| < \frac{\varepsilon}{2}$ (by 2.62). Now

$$|A \setminus B'| = |(A \setminus B) \cup (B \setminus B')| \leq |A \setminus B| + |B \setminus B'| < \varepsilon.$$

Because the restriction of a continuous function to a smaller domain is also continuous, $f|_{B'}$ is continuous, getting the desired result with closed sets.

Another simplification is that it suffices to prove the result (as modified by the paragraph above) in the case when the domain of f is all of $\mathbf{R}$. To see this, suppose we have indeed proved the result in that case. Suppose also that $f\colon A \to \mathbf{R}$ is Borel measurable and $\varepsilon > 0$. Extend f to a function $g\colon \mathbf{R} \to \mathbf{R}$ by defining

$$g(x) = \begin{cases} f(x) & \text{if } x \in A, \\ 0 & \text{if } x \in \mathbf{R} \setminus A. \end{cases}$$

Then g is Borel measurable, and thus there exists a Borel set $B \subset \mathbf{R}$ such that $|\mathbf{R} \setminus B| < \varepsilon$ and $g|_B$ is continuous. Let $B' = B \cap A$. Then $B' \subset A$ is a Borel set and $|A \setminus B'| \leq |\mathbf{R} \setminus B| < \varepsilon$. Furthermore, $f|_{B'} = g|_{B'}$, and thus $f|_{B'}$ is continuous, which completes the proof of this simplification.

Now consider the special case where $A = \mathbf{R}$ and $f = c_1\chi_{E_1} + \cdots + c_n\chi_{E_n}$ for some distinct nonzero $c_1, \ldots, c_n \in \mathbf{R}$ and some disjoint Borel sets $E_1, \ldots, E_n \subset \mathbf{R}$. Suppose $\varepsilon > 0$. For each $k \in \{1, \ldots, n\}$, there exist (by 2.68) a closed set $C_k \subset E_k$ and an open set $D_k \supset E_k$ such that

$$|D_k \setminus E_k| < \frac{\varepsilon}{2n} \quad \text{and} \quad |E_k \setminus C_k| < \frac{\varepsilon}{2n}.$$

Because $D_k \setminus C_k = (D_k \setminus E_k) \cup (E_k \setminus C_k)$, we have $|D_k \setminus C_k| < \frac{\varepsilon}{n}$ for each $k \in \{1, \ldots, n\}$.

Let

$$B = \Bigl(\bigcup_{k=1}^{n} C_k\Bigr) \cup \bigcap_{k=1}^{n} (\mathbf{R} \setminus D_k).$$

Then B is a closed subset of $\mathbf{R}$ and $\mathbf{R} \setminus B \subset \bigcup_{k=1}^{n} (D_k \setminus C_k)$. Thus $|\mathbf{R} \setminus B| < \varepsilon$.

Because $C_k \subset E_k$, we see that f is identically c_k on C_k. Thus $f|_{C_k}$ is continuous for each $k \in \{1, \ldots, n\}$. Because

$$\bigcap_{k=1}^{n} (\mathbf{R} \setminus D_k) \subset \bigcap_{k=1}^{n} (\mathbf{R} \setminus E_k),$$

we see that f is identically 0 on $\bigcap_{k=1}^{n} (\mathbf{R} \setminus D_k)$. Thus $f|_{\bigcap_{k=1}^{n} (\mathbf{R} \setminus D_k)}$ is continuous. Putting all this together, we conclude that $f|_B$ is continuous (use Exercise 9 in this section), completing the proof in this special case.

Now consider an arbitrary Borel measurable function $f \colon \mathbf{R} \to \mathbf{R}$. By 2.80, there exists a sequence $f_1, f_2, \ldots$ of functions from $\mathbf{R}$ to $\mathbf{R}$ that converges pointwise on $\mathbf{R}$ to f, where each f_n is a simple Borel measurable function.

Suppose $\varepsilon > 0$. By the special case already proved, for each $n \in \mathbf{Z}^+$, there exists a closed set $C_n \subset \mathbf{R}$ such that $|\mathbf{R} \setminus C_n| < \frac{\varepsilon}{2^{n+1}}$ and $f_n|_{C_n}$ is continuous. Let

$$C = \bigcap_{n=1}^{\infty} C_n.$$

Thus C is a closed set and $f_n|C$ is continuous for every $n \in \mathbf{Z}^+$. Note that $\mathbf{R} \setminus C = \bigcup_{n=1}^{\infty} (\mathbf{R} \setminus C_n)$; thus $|\mathbf{R} \setminus C| < \frac{\varepsilon}{2}$.

For each $m \in \mathbf{Z}$, the sequence $f_1|_{(m,m+1)}, f_2|_{(m,m+1)}, \ldots$ converges pointwise on $(m, m+1)$ to $f|_{(m,m+1)}$. Thus by Egorov's Theorem (2.77), for each $m \in \mathbf{Z}$, there is a Borel set $E_m \subset (m, m+1)$ such that $f_1, f_2, \ldots$ converges uniformly to f on E_m and

$$|(m, m+1) \setminus E_m| < \frac{\varepsilon}{2^{|m|+3}}.$$

Thus $f_1, f_2, \ldots$ converges uniformly to f on $C \cap E_m$ for each $m \in \mathbf{Z}$. Because each $f_n|_C$ is continuous, we conclude (using 2.76) that $f|_{C \cap E_m}$ is continuous for each $m \in \mathbf{Z}$. Thus $f|_B$ is continuous, where $B = \bigcup_{m \in \mathbf{Z}} (C \cap E_m)$. The proof is completed by noting that

$$\mathbf{R} \setminus B \subset \mathbf{Z} \cup \Bigl(\bigcup_{m \in \mathbf{Z}} \bigl((m, m+1) \setminus E_m\bigr)\Bigr) \cup (\mathbf{R} \setminus C),$$

and thus $|\mathbf{R} \setminus B| < \varepsilon$. ∎

We will need the following result to get another version of Luzin's Theorem.

2.82 *Continuous extensions of continuous functions*

- Every continuous function on a closed subset of $\mathbf{R}$ can be extended to a continuous function on all of $\mathbf{R}$.
- More precisely, if $B \subset \mathbf{R}$ is closed and $f\colon B \to \mathbf{R}$ is continuous, then there exists a continuous function $g\colon \mathbf{R} \to \mathbf{R}$ such that $g|_B = f$.

Proof Suppose $B \subset \mathbf{R}$ is closed and $f\colon B \to \mathbf{R}$ is continuous. Thus $\mathbf{R} \setminus B$ is the union of a collection of disjoint open intervals $\{I_n\}$. For each such interval of the form (a, ∞) or of the form $(-\infty, a)$ (there can be at most one interval of each form in $\{I_n\}$), define $g(x) = f(a)$ for all x in the interval.

For each interval I_n of the form (b, c) with $b < c$ and $b, c \in \mathbf{R}$, define g on $[b, c]$ to be the linear function such that $g(b) = f(b)$ and $g(c) = f(c)$.

Define $g(x) = f(x)$ for all $x \in \mathbf{R}$ for which $g(x)$ has not been defined by the previous two paragraphs. Then $g\colon \mathbf{R} \to \mathbf{R}$ is continuous and $g|_B = f$. ■

The next result gives a slightly modified way to state Luzin's Theorem. You can think of this version as saying that the value of a Borel measurable function can be changed on a set with small Lebesgue measure to produce a continuous function.

2.83 *Luzin's Theorem, second version*

Suppose $A \subset \mathbf{R}$ and $f\colon A \to \mathbf{R}$ is a Borel measurable function. Then for every $\varepsilon > 0$, there exists a closed set $B \subset A$ and a continuous function $g\colon \mathbf{R} \to \mathbf{R}$ such that $|A \setminus B| < \varepsilon$ and $g|_B = f|_B$.

Proof Suppose $\varepsilon > 0$. By the first version of Luzin's Theorem (2.81), there is a closed set $B \subset A$ such that $|A \setminus B| < \varepsilon$ and $f|_B$ is a continuous function on B. Use 2.82 to extend $f|_B$ to a continuous function $g\colon \mathbf{R} \to \mathbf{R}$. ■

A building at Moscow State University, which was founded in 1755. Egorov and Luzin were both students at Moscow State University. Both of them were later faculty members at the same institution. Luzin's PhD thesis advisor was Egorov.

Lebesgue Measurable Functions

2.84 Definition ***Lebesgue measurable function***

A function $f: A \to \mathbf{R}$, where $A \subset \mathbf{R}$, is called *Lebesgue measurable* if $f^{-1}(B)$ is a Lebesgue measurable set for every Borel set $B \subset \mathbf{R}$.

If $f: A \to \mathbf{R}$ is a Lebesgue measurable function, then A is a Lebesgue measurable subset of $\mathbf{R}$ [because $A = f^{-1}(\mathbf{R})$]. If A is a Lebesgue measurable subset of $\mathbf{R}$, then the definition above is the standard definition of an $\mathcal{S}$-measurable function, where $\mathcal{S}$ is a σ-algebra of all Lebesgue measurable subsets of A.

The following list summarizes and reviews some crucial definitions and results:

- A Borel set is an element of the smallest σ-algebra on $\mathbf{R}$ that contains all the open subsets of $\mathbf{R}$.
- A Lebesgue measurable set is an element of the smallest σ-algebra on $\mathbf{R}$ that contains all the open subsets of $\mathbf{R}$ and all the subsets of $\mathbf{R}$ with outer measure 0.
- The terminology *Lebesgue set* would make good sense in parallel to the terminology *Borel set*. However, *Lebesgue set* has another meaning, so we need to use *Lebesgue measurable set*.
- Every Lebesgue measurable set differs from a Borel set by a set with outer measure 0. The Borel set can be taken either to be contained in the Lebesgue measurable set or to contain the Lebesgue measurable set.
- Outer measure restricted to the σ-algebra of Borel sets is called Lebesgue measure.
- Outer measure restricted to the σ-algebra of Lebesgue measurable sets is also called Lebesgue measure.
- Outer measure is not a measure on the σ-algebra of all subsets of $\mathbf{R}$.
- A function $f: A \to \mathbf{R}$, where $A \subset \mathbf{R}$, is called Borel measurable if $f^{-1}(B)$ is a Borel set for every Borel set $B \subset \mathbf{R}$.
- A function $f: A \to \mathbf{R}$, where $A \subset \mathbf{R}$, is called Lebesgue measurable if $f^{-1}(B)$ is a Lebesgue measurable set for every Borel set $B \subset \mathbf{R}$.

Although there exist Lebesgue measurable sets that are not Borel sets, you will never run into one. Similarly, you will never run into a Lebesgue measurable function that is not Borel measurable. A great way to simplify the potential confusion about Lebesgue measurable functions being defined by inverse images of Borel sets is to consider only Borel measurable functions.

"Passing from Borel to Lebesgue measurable functions is the work of the devil. Don't even consider it!" –Barry Simon (winner of the American Mathematical Society Steele Prize for Lifetime Achievement), in his five-volume series A Comprehensive Course in Analysis

"He professes to have received no sinister measure."
–Measure for Measure,
by William Shakespeare

The next result states that if we adopt the philosophy that what happens on a set of outer measure 0 does not matter much, then we might as well restrict our attention to Borel measurable functions.

2.85 *Every Lebesgue measurable function is almost Borel measurable*

Suppose B is a Borel set and $f\colon B \to \mathbf{R}$ is a Lebesgue measurable function. Then there exists a Borel measurable function $g\colon B \to \mathbf{R}$ such that

$$|\{x \in B : g(x) \neq f(x)\}| = 0.$$

Proof There exists a sequence $f_1, f_2, \dots$ of Lebesgue measurable simple functions from B to $\mathbf{R}$ converging pointwise on B to f (by 2.80). Suppose $n \in \mathbf{Z}^+$. Then there exist $c_1, \dots, c_N \in \mathbf{R}$ and disjoint Lebesgue measurable sets $A_1, \dots, A_N \subset B$ such that

$$f_n = c_1\chi_{A_1} + \cdots + c_N\chi_{A_N}.$$

For each $k \in \{1, \dots, N\}$, there exists a Borel set $B_k \subset A_k$ such that $|A_k \setminus B_k| = 0$ [by the equivalence of (a) and (d) in 2.68]. Let

$$g_n = c_1\chi_{B_1} + \cdots + c_N\chi_{B_N}.$$

Then g_n is a Borel measurable function and $|\{x \in B : g_n(x) \neq f_n(x)\}| = 0$.

If $x \notin \bigcup_{n=1}^{\infty}\{x \in B : g_n(x) \neq f_n(x)\}$, then $g_n(x) = f_n(x)$ for all $n \in \mathbf{Z}^+$ and hence $\lim_{n\to\infty} g_n(x) = f(x)$. Let

$$E = \{x \in B : \lim_{n\to\infty} g_n(x) \text{ exists in } \mathbf{R}\}.$$

Then E is a Borel subset of B (by Exercise 15 in Section 2B). Also,

$$B \setminus E \subset \bigcup_{n=1}^{\infty}\{x \in B : g_n(x) \neq f_n(x)\}$$

and thus $|B \setminus E| = 0$. For $x \in B$, let

$$g(x) = \lim_{n\to\infty}(\chi_E g_n)(x).$$

If $x \in E$, then the limit above exists by the definition of E; if $x \in B \setminus E$, then the limit above exists because $(\chi_E g_n)(x) = 0$ for all $n \in \mathbf{Z}^+$.

For each $n \in \mathbf{Z}^+$, the function $\chi_E g_n$ is Borel measurable. Thus g is a Borel measurable function (by 2.45). Because

$$\{x \in B : g(x) \neq f(x)\} \subset \bigcup_{n=1}^{\infty}\{x \in B : g_n(x) \neq f_n(x)\},$$

we have $|\{x \in B : g(x) \neq f(x)\}| = 0$, completing the proof. ∎

EXERCISES 2E

1 Suppose X is a finite set. Explain why a sequence of functions from X to $\mathbf{R}$ that converges pointwise on X also converges uniformly on X.

2 Give an example of a sequence of functions from $\mathbf{Z}^+$ to $\mathbf{R}$ that converges pointwise on $\mathbf{Z}^+$ but does not converge uniformly on $\mathbf{Z}^+$.

3 Give an example of a sequence of continuous functions $f_1, f_2, \ldots$ from $[0,1]$ to $\mathbf{R}$ that converge pointwise to a function $f\colon [0,1] \to \mathbf{R}$ that is not a continuous function.

4 Prove or give a counterexample: If $A \subset \mathbf{R}$ and $f_1, f_2, \ldots$ is a sequence of uniformly continuous functions from A to $\mathbf{R}$ that converge uniformly to a function $f\colon A \to \mathbf{R}$, then f is uniformly continuous on A.

5 Give an example to show that Egorov's Theorem can fail without the hypothesis that $\mu(X) < \infty$.

6 Suppose $(X, \mathcal{S}, \mu)$ is a measure space with $\mu(X) < \infty$. Suppose $f_1, f_2, \ldots$ is a sequence of $\mathcal{S}$-measurable functions from X to $\mathbf{R}$ such that $\lim_{n\to\infty} f_n(x) = \infty$ for each $x \in X$. Prove that for every $\varepsilon > 0$, there exists a set $E \in \mathcal{S}$ such that $\mu(X \setminus E) < \varepsilon$ and $f_1, f_2, \ldots$ converges uniformly to ∞ on E (meaning that for every $t > 0$, there exists $N \in \mathbf{Z}^+$ such that $f_n(x) > t$ for all integers $n \geq N$ and all $x \in E$).

[*The exercise above is an Egorov-type theorem for sequences of functions that converge pointwise to* ∞.]

7 Suppose E is a closed bounded subset of $\mathbf{R}$ and $f_1, f_2, \ldots$ is an increasing sequence of continuous real-valued functions on E (thus $f_1(x) \leq f_2(x) \leq \cdots$ for all $x \in E$) such that $\sup\{f_1(x), f_2(x), \ldots\} < \infty$ for each $x \in E$. Define a real-valued function f on E by

$$f(x) = \lim_{n\to\infty} f_n(x).$$

Prove that f is continuous on E if and only if $f_1, f_2, \ldots$ converges uniformly on E to f.

[*The result above is called Dini's Theorem.*]

8 Suppose μ is the measure on $(\mathbf{Z}^+, 2^{\mathbf{Z}^+})$ defined by

$$\mu(E) = \sum_{n\in E} \frac{1}{2^n}.$$

Prove that for every $\varepsilon > 0$, there exists a set $E \subset \mathbf{Z}^+$ with $\mu(\mathbf{Z}^+ \setminus E) < \varepsilon$ such that $f_1, f_2, \ldots$ converges uniformly on E for every sequence of functions $f_1, f_2, \ldots$ from $\mathbf{Z}^+$ to $\mathbf{R}$ that converges pointwise on $\mathbf{Z}^+$.

[*This result does not follow from Egorov's Theorem because here we are asking for E to depend only on* ε. *In Egorov's Theorem, E depends on* ε *and on the sequence* $f_1, f_2, \ldots$.]

9 Suppose $C_1, \dots, C_n$ are disjoint closed subsets of $\mathbf{R}$. Prove that if

$$f \colon C_1 \cup \cdots \cup C_n \to \mathbf{R}$$

is a function such that $f|_{C_k}$ is a continuous function for each $k \in \{1, \dots, n\}$, then f is a continuous function.

10 Suppose $B \subset \mathbf{R}$ is such that every continuous function from B to $\mathbf{R}$ can be extended to a continuous function from $\mathbf{R}$ to $\mathbf{R}$. Prove that B is a closed subset of $\mathbf{R}$.

11 Prove or give a counterexample: If $B \subset \mathbf{R}$ is such that every bounded continuous function from B to $\mathbf{R}$ can be extended to a continuous function from $\mathbf{R}$ to $\mathbf{R}$, then B is a closed subset of $\mathbf{R}$.

12 Give an example of a Borel measurable function from $\mathbf{R}$ to $\mathbf{R}$ that cannot be modified on a set of Lebesgue measure 0 to become a continuous function.

13 Prove or give a counterexample: If $f_t \colon \mathbf{R} \to \mathbf{R}$ is a Borel measurable function for each $t \in \mathbf{R}$ and $f \colon \mathbf{R} \to (-\infty, \infty]$ is defined by

$$f(x) = \sup\{f_t(x) : t \in \mathbf{R}\},$$

then f is a Borel measurable function.