

Contents

Preface for Instructors xi

Preface for Students xv

Acknowledgments xvi

1 Review: The Real Numbers 1

1A Complete Ordered Fields 2

Fields 2

Ordered Fields 3

Completeness 7

Exercises 1A 10

1B Construction of the Real Numbers: Dedekind Cuts 12

Exercises 1B 16

1C Sup and Inf 17

Archimedean Property 17

Supremum and Infimum 18

Irrational Numbers 20

Intervals 21

Exercises 1C 22

1D Open and Closed Subsets of \mathbb{R} 25

Limits 25

Open Subsets 26

Closed Subsets 29

Exercises 1D 31

1E Sequences and Continuity 34

The Bolzano–Weierstrass Theorem 34

Continuity and Uniform Continuity 37

Max and Min on Closed Bounded Sets 39

Exercises 1E 40

| | | |
|----------|--|-----------|
| 2 | <i>Review: Riemann Integration</i> | 43 |
| 2A | The Riemann Integral | 44 |
| | Exercises 2A | 49 |
| 2B | Why the Riemann Integral is Not Good Enough | 51 |
| | Exercises 2B | 54 |
| 3 | <i>Measures</i> | 55 |
| 3A | Outer Measure on \mathbf{R} | 56 |
| | Motivation and Definition of Outer Measure | 56 |
| | Good Properties of Outer Measure | 57 |
| | The Outer Measure of a Closed Bounded Interval | 60 |
| | Outer Measure is Not Additive | 63 |
| | Exercises 3A | 65 |
| 3B | Measurable Spaces and Functions | 67 |
| | σ -Algebras | 68 |
| | The Borel Subsets of \mathbf{R} | 70 |
| | Inverse Images | 71 |
| | Measurable Functions | 73 |
| | Exercises 3B | 80 |
| 3C | Measures and Their Properties | 83 |
| | Definition and Examples of Measures | 83 |
| | Properties of Measures | 84 |
| | Exercises 3C | 87 |
| 3D | Lebesgue Measure | 89 |
| | Additivity of outer measure on Borel sets | 89 |
| | Lebesgue Measurable Sets | 94 |
| | The Cantor Set | 97 |
| | Exercises 3D | 100 |
| 3E | Functions on Measure Spaces | 102 |
| | Pointwise and Uniform Convergence | 102 |
| | Egorov's Theorem | 103 |
| | Simple Functions | 104 |
| | Luzin's Theorem | 106 |
| | Lebesgue Measurable Functions | 109 |
| | Exercises 3E | 111 |

4 Integration 113**4A Integration with Respect to a Measure 114**Integration of Nonnegative Functions **114**Monotone Convergence Theorem **118**Integration of Real-Valued Functions **120**Exercises 4A **123****4B Interchanging Limits and Integrals 125**Integration on Subsets **125**Almost Everywhere **127**Dominated Convergence Theorem **127**Riemann Integrals and Lebesgue Integrals **129**Exercises 4B **130**