

# *Measure, Integration & Real Analysis*

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preliminary edition  
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*Dedicated to*

*Paul Halmos, Don Sarason, and Allen Shields,*

*the three mathematicians who most  
helped me become a mathematician.*

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