

Measure, Integration & Real Analysis

preliminary edition
18 July 2018

Sheldon Axler

Dedicated to

Paul Halmos, Don Sarason, and Allen Shields,

*the three mathematicians who most
helped me become a mathematician.*

Contents

<i>Preface for Instructors</i>	xii
<i>Preface for Students</i>	xvii
<i>Acknowledgments</i>	xviii
1 Riemann Integration	1
1A Review: Riemann Integral	2
Exercises 1A	7
1B Why the Riemann Integral is Not Good Enough	9
Exercises 1B	12
2 Measures	13
2A Outer Measure on \mathbf{R}	14
Motivation and Definition of Outer Measure	14
Good Properties of Outer Measure	15
Outer Measure of a Closed Bounded Interval	18
Outer Measure is Not Additive	21
Exercises 2A	23
2B Measurable Spaces and Functions	25
σ -Algebras	26
Borel Subsets of \mathbf{R}	28
Inverse Images	29
Measurable Functions	31
Exercises 2B	38
2C Measures and Their Properties	41
Definition and Examples of Measures	41
Properties of Measures	42
Exercises 2C	45

2D	Lebesgue Measure	47
	Additivity of outer measure on Borel sets	47
	Lebesgue Measurable Sets	52
	Cantor Set	55
	Exercises 2D	58
2E	Functions on Measure Spaces	60
	Pointwise and Uniform Convergence	60
	Egorov's Theorem	61
	Approximation by Simple Functions	63
	Luzin's Theorem	64
	Lebesgue Measurable Functions	67
	Exercises 2E	69
3	Integration	71
3A	Integration with Respect to a Measure	72
	Integration of Nonnegative Functions	72
	Monotone Convergence Theorem	78
	Integration of Real-Valued Functions	80
	Exercises 3A	83
3B	Limits of Integrals & Integrals of Limits	85
	Bounded Convergence Theorem	85
	Sets of Measure 0 in Integration Theorems	86
	Dominated Convergence Theorem	87
	Riemann Integrals and Lebesgue Integrals	90
	Approximation by Nice Functions	92
	Exercises 3B	96
4	Differentiation	98
4A	The Hardy–Littlewood Maximal Function	99
	Markov's Inequality	99
	Vitali Covering Lemma	100
	Hardy–Littlewood Maximal Inequality	101
	Exercises 4A	103
4B	Derivatives of Integrals	105
	Lebesgue Differentiation Theorem	105
	Derivatives	107
	Density	109
	Exercises 4B	112

5 Product Measures 114**5A Products of Measure Spaces 115**

- Products of σ -Algebras 115
- Monotone Class Theorem 118
- Products of Measures 121
- Exercises 5A 126

5B Iterated Integrals 127

- Tonelli's Theorem 127
- Fubini's Theorem 129
- Area Under the Graph of a Function 131
- Exercises 5B 133

5C Lebesgue Integration on \mathbf{R}^N 134

- Borel Subsets of \mathbf{R}^N 134
- Lebesgue Measure on \mathbf{R}^N 137
- The Volume of the Unit Ball in \mathbf{R}^N 138
- Equality of Mixed Partial Derivatives Via Fubini's Theorem 140
- Exercises 5C 142

6 Banach Spaces 144**6A Vector Spaces 145**

- Integration of Complex-Valued Functions 145
- Vector Spaces and Subspaces 148
- Exercises 6A 151

6B Normed Vector Spaces 152

- Norms and Cauchy Sequences 152
- Open Sets, Closed Sets, and Continuity 156
- Bounded Linear Maps 159
- Linear Functionals 162
- Exercises 6B 164

7 L^p Spaces 167**7A $\mathcal{L}^p(\mu)$ 168**

- Hölder's Inequality 168
- Minkowski's Inequality 172
- Exercises 7A 173

7B $L^p(\mu)$ 176

- Definition of $L^p(\mu)$ 176
- $L^p(\mu)$ is a Banach Space 178
- Duality 180
- Exercises 7B 182

8 *Hilbert Spaces* 183

8A Inner Product Spaces 184

- Inner Products 184
- Cauchy–Schwarz Inequality and Triangle Inequality 187
- Exercises 8A 194

8B Orthogonality 197

- Orthogonal Projections 197
- Orthogonal Complements 199
- Orthonormal Bases 199
- Exercises 8B 199

8C Linear Maps on Hilbert Spaces 203

- Riesz Representation Theorem 203
- Adjoint of a Linear Map 203
- Compact Operators 203
- Spectral Theorem for Compact Normal Operators 203
- Exercises 8C 203

9 *Fourier Analysis* 204

9A Fourier Series 205

9B Fourier Transforms 206

- Exercises 9B 206

10 *Signed and Complex Measures* 207

10A Dual of $C(K)$ 208

10B The Cantor Function 209

- Exercises 10B 209

10C Absolute Continuity 213

- Integrals of Derivatives 213
- Radon–Nikodym Theorem 213
- Functions of Bounded Variation 213

10D Lebesgue–Stieltjes Integration	214
Exercises 10D	214
11 Probability Measures	215
Exercises 11	216
0 Appendix: The Real Numbers and \mathbf{R}^N	217
A Complete Ordered Fields	218
Fields	218
Ordered Fields	219
Completeness	223
Exercises A	226
B Construction of the Real Numbers: Dedekind Cuts	228
Exercises B	232
C Supremum and Infimum	233
Archimedean Property	233
Greatest Lower Bound	234
Irrational Numbers	236
Intervals	237
Exercises C	238
D Open and Closed Subsets of \mathbf{R}^N	241
Limits in \mathbf{R}^N	241
Open Subsets of \mathbf{R}^N	243
Closed Subsets of \mathbf{R}^N	246
Exercises D	249
E Sequences and Continuity	251
Bolzano–Weierstrass Theorem	251
Continuity and Uniform Continuity	254
Max and Min on Closed Bounded Subsets of \mathbf{R}^N	256
Exercises E	257
Photo Credits	260