

Measure, Integration & Real Analysis

preliminary edition
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Dedicated to

Paul Halmos, Don Sarason, and Allen Shields,

*the three mathematicians who most
helped me become a mathematician.*

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