

Measure, Integration & Real Analysis

preliminary edition
14 October 2018

Sheldon Axler

Dedicated to

Paul Halmos, Don Sarason, and Allen Shields,

*the three mathematicians who most
helped me become a mathematician.*

Contents *(tentative beyond Chapter 7)*

***Preface for Students* xiii**

***Acknowledgments* xiv**

1 Riemann Integration 1

1A Review: Riemann Integral 2

Exercises 1A 7

1B Why the Riemann Integral is Not Good Enough 9

Exercises 1B 12

2 Measures 13

2A Outer Measure on \mathbf{R} 14

Motivation and Definition of Outer Measure 14

Good Properties of Outer Measure 15

Outer Measure of a Closed Bounded Interval 18

Outer Measure is Not Additive 21

Exercises 2A 23

2B Measurable Spaces and Functions 25

σ -Algebras 26

Borel Subsets of \mathbf{R} 28

Inverse Images 29

Measurable Functions 31

Exercises 2B 38

2C Measures and Their Properties 41

Definition and Examples of Measures 41

Properties of Measures 42

Exercises 2C 45

2D Lebesgue Measure 47

Additivity of Outer Measure on Borel Sets 47

Lebesgue Measurable Sets 52

Cantor Set 55
Exercises 2D 58

2E Functions on Measure Spaces 60
Pointwise and Uniform Convergence 60
Egorov's Theorem 61
Approximation by Simple Functions 63
Luzin's Theorem 64
Lebesgue Measurable Functions 67
Exercises 2E 69

3 Integration 71

3A Integration with Respect to a Measure 72
Integration of Nonnegative Functions 72
Monotone Convergence Theorem 78
Integration of Real-Valued Functions 80
Exercises 3A 83
3B Limits of Integrals & Integrals of Limits 85
Bounded Convergence Theorem 85
Sets of Measure 0 in Integration Theorems 86
Dominated Convergence Theorem 87
Riemann Integrals and Lebesgue Integrals 90
Approximation by Nice Functions 92
Exercises 3B 96

4 Differentiation 98

4A The Hardy–Littlewood Maximal Function 99
Markov's Inequality 99
Vitali Covering Lemma 100
Hardy–Littlewood Maximal Inequality 101
Exercises 4A 103
4B Derivatives of Integrals 105
Lebesgue Differentiation Theorem 105
Derivatives 107
Density 109
Exercises 4B 112

5 Product Measures 114**5A Products of Measure Spaces 115**

- Products of σ -Algebras 115
- Monotone Class Theorem 118
- Products of Measures 121
- Exercises 5A 126

5B Iterated Integrals 127

- Tonelli's Theorem 127
- Fubini's Theorem 129
- Area Under the Graph of a Function 131
- Exercises 5B 133

5C Lebesgue Integration on \mathbf{R}^N 134

- Borel Subsets of \mathbf{R}^N 134
- Lebesgue Measure on \mathbf{R}^N 137
- The Volume of the Unit Ball in \mathbf{R}^N 138
- Equality of Mixed Partial Derivatives Via Fubini's Theorem 140
- Exercises 5C 142

6 Banach Spaces 144**6A Vector Spaces 145**

- Integration of Complex-Valued Functions 145
- Vector Spaces and Subspaces 148
- Exercises 6A 151

6B Normed Vector Spaces 152

- Norms and Cauchy Sequences 152
- Open Sets, Closed Sets, and Continuity 156
- Bounded Linear Maps 159
- Linear Functionals 162
- Exercises 6B 164

7 L^p Spaces 168**7A $\mathcal{L}^p(\mu)$ 169**

- Hölder's Inequality 169
- Minkowski's Inequality 173
- Exercises 7A 174

7B $L^p(\mu)$ 177

Definition of $L^p(\mu)$ 177

$L^p(\mu)$ is a Banach Space 179

Duality 181

Exercises 7B 183

8 *Hilbert Spaces* 185

8A Inner Product Spaces 186

Inner Products 186

Cauchy–Schwarz Inequality and Triangle Inequality 189

Exercises 8A 196

8B Orthogonality 199

Orthogonal Projections 199

Orthogonal Complements 204

Riesz Representation Theorem 208

Exercises 8B 209

8C Orthonormal Bases 212

Bessel’s Inequality 212

Parseval’s Identity 219

The Gram–Schmidt Process 221

Existence of Orthonormal Bases 224

Riesz Representation Theorem, Revisited 226

Exercises 8C 227

9 *Linear Maps on Hilbert Spaces* 231

9A Adjoints 232

9B Compact Operators 233

9C Spectral Theorem for Compact Normal Operators 234

Exercises 9C 234

10 *Fourier Analysis* 235

10A Fourier Series 236

10B Fourier Transforms 237

Exercises 10B 237

11	<i>Signed and Complex Measures</i>	238
11A	Dual of $C(K)$	239
11B	The Cantor Function	240
	Exercises 11B	241
11C	Absolute Continuity	244
	Integrals of Derivatives	244
	Radon–Nikodym Theorem	244
	Functions of Bounded Variation	244
11D	Lebesgue–Stieltjes Integration	245
	Exercises 11D	245
12	<i>Probability Measures</i>	246
	Exercises 12	247
0	<i>Appendix: The Real Numbers and \mathbf{R}^N</i>	248
A	Complete Ordered Fields	249
	Fields	249
	Ordered Fields	250
	Completeness	254
	Exercises A	258
B	Construction of the Real Numbers: Dedekind Cuts	260
	Exercises B	264
C	Supremum and Infimum	265
	Archimedean Property	265
	Greatest Lower Bound	266
	Irrational Numbers	268
	Intervals	269
	Exercises C	270
D	Open and Closed Subsets of \mathbf{R}^N	273
	Limits in \mathbf{R}^N	273
	Open Subsets of \mathbf{R}^N	275
	Closed Subsets of \mathbf{R}^N	278
	Exercises D	281

E Sequences and Continuity 283

Bolzano–Weierstrass Theorem **283**

Continuity and Uniform Continuity **286**

Max and Min on Closed Bounded Subsets of \mathbf{R}^N **288**

Exercises E **289**

***Photo Credits* 292**