Chapter
9

# Real and Complex Measures

A measure is a countably additive function from a $\sigma$-algebra to $[0, \infty]$. In this chapter, we consider countably additive functions from a $\sigma$-algebra to either **R** or **C**. The first section of this chapter shows that these functions, called real measures or complex measures, form an interesting Banach space with an appropriate norm.

The second section of this chapter focuses on decomposition theorems that help us understand real and complex measures. These results will lead to a proof that the dual space of $L^p(\mu)$ can be identified with $L^{p'}(\mu)$.

*Dome in the main building of the University of Vienna, where Johann Radon (1887–1956) was a student and then later a faculty member. The Radon–Nikodym Theorem provides information analogous to differentiation for measures.*

# 9A Total Variation

## Properties of Real and Complex Measures

Recall that a measurable space is a pair $(X, \mathcal{S})$, where $\mathcal{S}$ is a $\sigma$-algebra on $X$. Recall also that a measure on $(X, \mathcal{S})$ is a countably additive function from $\mathcal{S}$ to $[0, \infty]$ that takes $\emptyset$ to 0. Countably additive functions that take values in $\mathbf{R}$ or $\mathbf{C}$ give us new objects called real measures or complex measures.

**9.1 Definition** ***real and complex measures***

Suppose $(X, \mathcal{S})$ is measurable space.

- A function $\nu \colon \mathcal{S} \to \mathbf{F}$ is called *countably additive* if

$$\nu\Big(\bigcup_{k=1}^{\infty} E_k\Big) = \sum_{k=1}^{\infty} \nu(E_k)$$

  for every disjoint sequence $E_1, E_2, \dots$ of sets in $\mathcal{S}$.

- A *real measure* on $(X, \mathcal{S})$ is a countably additive function $\nu \colon \mathcal{S} \to \mathbf{R}$.
- A *complex measure* on $(X, \mathcal{S})$ is a countably additive function $\nu \colon \mathcal{S} \to \mathbf{C}$.

*The terminology* nonnegative measure *would be more appropriate than* positive measure *because the function $\mu \colon \mathcal{S} \to \mathbf{F}$ defined by $\mu(E) = 0$ for every $E \in \mathcal{S}$ is a positive measure. However, we will stick with tradition and use the phrase* positive measure.

The word *measure* can be ambiguous in the mathematical literature. The most common use of the word *measure* is as we defined it in Chapter 2 (see 2.53). However, some mathematicians use the word *measure* to include what are here called real and complex measures; they then use the phrase *positive measure* to refer to what we defined as a measure in 2.53. To help relieve this ambiguity, this chapter will usually use the phrase (*positive*) *measure* to refer to measures as defined in 2.53. Putting *positive* in parentheses helps reinforce the idea that it is optional while distinguishing such measures from real and complex measures.

**9.2 Example** *real and complex measures*

- Let $\lambda$ denote Lebesgue measure on $[-1, 1]$. Define $\nu$ on the Borel subsets of $[-1, 1]$ by

$$\nu(E) = \lambda\big(E \cap [0, 1]\big) - \lambda\big(E \cap [-1, 0)\big).$$

  Then $\nu$ is a real measure.

- If $\mu_1$ and $\mu_2$ are finite (positive) measures, then $\mu_1 - \mu_2$ is a real measure and $\alpha_1 \mu_1 + \alpha_2 \mu_2$ is a complex measure for all $\alpha_1, \alpha_2 \in \mathbf{C}$.
- If $\nu$ is a complex measure, then $\operatorname{Re} \nu$ and $\operatorname{Im} \nu$ are real measures.

Note that every real measure is a complex measure. Note also that by definition, $\infty$ is not an allowable value for a real or complex measure. Thus a (positive) measure $\mu$ on $(X, \mathcal{S})$ is a real measure if and only if $\mu(X) < \infty$.

Some authors use the terminology *signed measure* instead of *real measure*; some authors allow a real measure to take on the value $\infty$ or $-\infty$ (but not both, because the expression $\infty - \infty$ must be avoided). However, real measures as defined here will serve us better because we need to avoid $\pm\infty$ when considering the Banach space of real or complex measures on a measurable space (see 9.18).

For (positive) measures, we had to make $\mu(\emptyset) = 0$ part of the definition to avoid the function $\mu$ that assigns $\infty$ to all sets, including the empty set. But $\infty$ is not an allowable value for real or complex measures. Thus $\nu(\emptyset) = 0$ is a consequence of our definition rather than part of the definition, as shown in the next result.

**9.3 *Absolute convergence for a disjoint union***

Suppose $\nu$ is a complex measure on a measurable space $(X, \mathcal{S})$. Then

(a) $\nu(\emptyset) = 0$;

(b) $\displaystyle\sum_{k=1}^{\infty} |\nu(E_k)| < \infty$ for every disjoint sequence $E_1, E_2, \dots$ of sets in $\mathcal{S}$.

Proof To prove (a), note that $\emptyset, \emptyset, \dots$ is a disjoint sequence of sets in $\mathcal{S}$ whose union equals $\emptyset$. Thus

$$\nu(\emptyset) = \sum_{k=1}^{\infty} \nu(\emptyset).$$

The right side of the equation above makes sense as an element of $\mathbf{R}$ or $\mathbf{C}$ only when $\mu(\emptyset) = 0$, which proves (a).

To prove (b), suppose $E_1, E_2, \dots$ is a disjoint sequence of sets in $\mathcal{S}$. First suppose $\nu$ is a real measure. Thus

$$\nu\Big(\bigcup_{\{k : \nu(E_k) > 0\}} E_k\Big) = \sum_{\{k : \nu(E_k) > 0\}} \nu(E_k) = \sum_{\{k : \nu(E_k) > 0\}} |\nu(E_k)|$$

and

$$-\nu\Big(\bigcup_{\{k : \nu(E_k) < 0\}} E_k\Big) = -\sum_{\{k : \nu(E_k) < 0\}} \nu(E_k) = \sum_{\{k : \nu(E_k) < 0\}} |\nu(E_k)|.$$

Because $\nu(E) \in \mathbf{R}$ for every $E \in \mathcal{S}$, the right side of the last two displayed equations is finite. Thus $\sum_{k=1}^{\infty} |\nu(E_k)| < \infty$, as desired.

Now consider the case where $\nu$ is a complex measure. Then

$$\sum_{k=1}^{\infty} |\nu(E_k)| \le \sum_{k=1}^{\infty} \big(|(\operatorname{Re}\nu)(E_k)| + |(\operatorname{Im}\nu)(E_k)|\big) < \infty,$$

where the last inequality follows from applying the result for real measures to the real measures $\operatorname{Re}\nu$ and $\operatorname{Im}\nu$. ■

The next definition provides an important class of examples of real and complex measures. If $\mathbf{F} = \mathbf{R}$, then the measure $\nu$ in the next result is a real measure (which is also a complex measure).

**9.4 *Measure determined by an $\mathcal{L}^1$-function***

Suppose $\mu$ is a (positive) measure on a measurable space $(X, \mathcal{S})$ and $h \in \mathcal{L}^1(\mu)$. Define $\nu\colon \mathcal{S} \to \mathbf{C}$ by

$$\nu(E) = \int_E h\, d\mu.$$

Then $\nu$ is a complex measure on $(X, \mathcal{S})$.

Proof Suppose $E_1, E_2, \dots$ is a disjoint sequence of sets in $\mathcal{S}$. Then

9.5
$$\nu\Bigl(\bigcup_{k=1}^{\infty} E_k\Bigr) = \int \Bigl(\sum_{k=1}^{\infty} \chi_{E_k}(x)h(x)\Bigr)\, d\mu(x) = \sum_{k=1}^{\infty} \int \chi_{E_k} h\, d\mu = \sum_{k=1}^{\infty} \nu(E_k),$$

where the first equality holds because the sets $E_1, E_2, \dots$ are disjoint and the second equality follows from the inequality

$$\Bigl|\sum_{k=1}^{m} \chi_{E_k}(x)h(x)\Bigr| \le |h(x)|,$$

which along with the assumption that $h \in \mathcal{L}^1(\mu)$ allows us to interchange the integral and limit of the partial sums by the Dominated Convergence Theorem (3.30).

The countably additivity shown in 9.5 means $\nu$ is a complex measure. ■

In the notation that we are about to define, the symbol $d$ has no separate meaning—it functions to separate $h$ and $\mu$. The result above shows that $h\, d\mu$ as defined below is indeed a real or complex measure.

**9.6 Definition** $h\, d\mu$

Suppose $\mu$ is a (positive) measure on a measurable space $(X, \mathcal{S})$ and $h \in \mathcal{L}^1(\mu)$. Then $h\, d\mu$ is the real or complex measure on $(X, \mathcal{S})$ defined by

$$(h\, d\mu)(E) = \int_E h\, d\mu.$$

Note that if a function $h \in \mathcal{L}^1(\mu)$ takes values in $[0, \infty)$, then $h\, d\mu$ is a finite (positive) measure.

The next result shows some basic properties of complex measures. No proofs are given because the proofs are the same as the proofs of the corresponding results for (positive) measures. Specifically, see the proofs of 2.56, 2.60, 2.58, and 2.59. Because complex measures cannot take on the value $\infty$, we do not need to worry about hypotheses of finite measure that are required of the (positive) measure versions of all but part (c).

**9.7 *Properties of complex measures***

Suppose $\nu$ is a complex measure on a measurable space $(X, \mathcal{S})$. Then

(a) $\nu(E \setminus D) = \nu(E) - \nu(D)$ for all $D, E \in \mathcal{S}$ with $D \subset E$;

(b) $\nu(D \cup E) = \nu(D) + \nu(E) - \nu(D \cap E)$ for all $D, E \in \mathcal{S}$;

(c) $\nu\Bigl(\bigcup_{k=1}^{\infty} E_k\Bigr) = \lim_{k\to\infty} \nu(E_k)$

for all increasing sequences $E_1 \subset E_2 \subset \cdots$ of sets in $\mathcal{S}$;

(d) $\nu\Bigl(\bigcap_{k=1}^{\infty} E_k\Bigr) = \lim_{k\to\infty} \nu(E_k)$

for all decreasing sequences $E_1 \supset E_2 \supset \cdots$ of sets in $\mathcal{S}$.

## Total Variation Measure

We use the terminology *total variation measure* below even though the object being defined is not obviously is a measure. Soon we will justify this terminology (see 9.11).

**9.8 Definition *total variation measure***

Suppose $\nu$ is a complex measure on a measurable space $(X, \mathcal{S})$. The *total variation measure* is the function $|\nu|: \mathcal{S} \to [0, \infty]$ defined by

$$|\nu|(E) = \sup\bigl\{|\nu(E_1)| + \cdots + |\nu(E_n)| : n \in \mathbf{Z}^+ \text{ and } E_1, \dots, E_n \text{ are disjoint sets in } \mathcal{S} \text{ such that } E_1 \cup \cdots \cup E_n \subset E\bigr\}.$$

To start getting familiar with the definition above, you should verify that if $\nu$ is a complex measure on $(X, \mathcal{S})$ and $E \in \mathcal{S}$, then

- $|\nu(E)| \leq |\nu|(E)$;
- $|\nu|(E) = \nu(E)$ if $\nu$ is a finite (positive) measure;
- $|\nu|(E) = 0$ if and only if $\nu(A) = 0$ for every $A \in \mathcal{S}$ such that $A \subset E$.

The next result states that for real measures, we can consider only $n = 2$ in the definition of the total variation measure.

**9.9 *Total variation measure of a real measure***

Suppose $\nu$ is a real measure on a measurable space $(X, \mathcal{S})$ and $E \in \mathcal{S}$. Then

$$|\nu|(E) = \sup\{|\nu(A)| + |\nu(B)| : A, B \text{ are disjoint sets in } \mathcal{S} \text{ and } A \cup B \subset E\}.$$

**Proof** Suppose that $n \in \mathbf{Z}^+$ and $E_1, \dots, E_n$ are disjoint sets in $\mathcal{S}$ such that $E_1 \cup \cdots \cup E_n \subset E$. Let

$$A = \bigcup_{\{k : \nu(E_k) > 0\}} E_k \quad \text{and} \quad B = \bigcup_{\{k : \nu(E_k) < 0\}} E_k.$$

Then $A, B$ are disjoint sets in $\mathcal{S}$ and $A \cup B \subset E$. Furthermore,

$$|\nu(A)| + |\nu(B)| = |\nu(E_1)| + \cdots + |\nu(E_n)|.$$

Thus in the supremum that defines $|\nu|(E)$, we can take $n = 2$. ■

The next result could be rephrased as stating that if $h \in \mathcal{L}^1(\mu)$, then the total variation measure of the measure $h\,d\mu$ is the measure $|h|\,d\mu$. In the statement below, the notation $d\nu = h\,d\mu$ means the same as $\nu = h\,d\mu$; the notation $d\nu$ is commonly used when considering expressions involving measures of the form $h\,d\mu$.

**9.10 *Total variation measure of* $h\,d\mu$**

Suppose $\mu$ is a (positive) measure on a measurable space $(X, \mathcal{S})$, $h \in \mathcal{L}^1(\mu)$, and $d\nu = h\,d\mu$. Then

$$|\nu|(E) = \int_E |h|\,d\mu$$

for every $E \in \mathcal{S}$.

**Proof** Suppose that $E \in S$. If $E_1, \dots, E_n$ is a disjoint sequence in $\mathcal{S}$ such that $E_1 \cup \cdots \cup E_n \subset E$, then

$$\sum_{k=1}^n |\nu(E_k)| = \sum_{k=1}^n \left| \int_{E_k} h\,d\mu \right| \leq \sum_{k=1}^n \int_{E_k} |h|\,d\mu \leq \int_E |h|\,d\mu.$$

The inequality above implies that $|\nu|(E) \leq \int_E |h|\,d\mu$.

To prove the inequality in the other direction, first suppose that $\mathbf{F} = \mathbf{R}$; thus $h$ is a real-valued function and $\nu$ is a real measure. Let

$$A = \{x \in E : h(x) > 0\} \quad \text{and} \quad B = \{x \in E : h(x) < 0\}.$$

Then $A$ and $B$ are disjoint sets in $\mathcal{S}$ and $A \cup B \subset E$. We have

$$|\nu(A)| + |\nu(B)| = \int_A h\,d\mu - \int_B h\,d\mu = \int_E |h|\,d\mu.$$

Thus $|\nu|(E) \geq \int_E |h|\,d\mu$, completing the proof in the case $\mathbf{F} = \mathbf{R}$.

Now suppose $\mathbf{F} = \mathbf{C}$; thus $\nu$ is a complex measure. Let $\varepsilon > 0$. There exists a simple function $g \in \mathcal{L}^1(\mu)$ such that $\|g - h\|_1 < \varepsilon$ (by 3.43). There exist disjoint sets $E_1, \dots, E_n \in \mathcal{S}$ and $c_1, \dots, c_n \in \mathbf{C}$ such that $E_1 \cup \cdots \cup E_n \subset E$ and

$$g|_E = \sum_{k=1}^n c_k \chi_{E_k}.$$

Now

$$\begin{aligned}\sum_{k=1}^{n}|\nu(E_k)| &= \sum_{k=1}^{n}\Bigl|\int_{E_k} h\,d\mu\Bigr|\\ &\geq \sum_{k=1}^{n}\Bigl|\int_{E_k} g\,d\mu\Bigr| - \sum_{k=1}^{n}\Bigl|\int_{E_k}(g-h)\,d\mu\Bigr|\\ &= \sum_{k=1}^{n}|c_k|\mu(E_k) - \sum_{k=1}^{n}\Bigl|\int_{E_k}(g-h)\,d\mu\Bigr|\\ &= \int_E |g|\,d\mu - \sum_{k=1}^{n}\Bigl|\int_{E_k}(g-h)\,d\mu\Bigr|\\ &\geq \int_E |g|\,d\mu - \sum_{k=1}^{n}\int_{E_k}|g-h|\,d\mu\\ &\geq \int_E |h|\,d\mu - 2\varepsilon.\end{aligned}$$

The inequality above implies that $|\nu|(E) \geq \int_E |h|\,d\mu - 2\varepsilon$. Because $\varepsilon$ is an arbitrary positive number, this implies $|\nu|(E) \geq \int_E |h|\,d\mu$, completing the proof. ■

Now we justify the terminology *total variation measure*.

**9.11 *Total variation measure is a measure***

Suppose $\nu$ is a complex measure on a measurable space $(X, \mathcal{S})$. Then the total variation function $|\nu|$ is a (positive) measure on $(X, \mathcal{S})$.

**Proof** The definition of $|\nu|$ and 9.3(a) imply that $|\nu|(\varnothing) = 0$.

To show that $|\nu|$ is countably additive, suppose $A_1, A_2, \ldots$ are disjoint sets in $\mathcal{S}$. Fix $m \in \mathbf{Z}^+$. For each $k \in \{1, \ldots, m\}$, suppose $E_{1,k}, \ldots, E_{n_k,k}$ are disjoint sets in $\mathcal{S}$ such that

**9.12** $$E_{1,k} \cup \ldots \cup E_{n_k,k} \subset A_k.$$

Then $\{E_{j,k} : 1 \leq k \leq m \text{ and } 1 \leq j \leq n_k\}$ is a disjoint collection of sets in $\mathcal{S}$ that are all contained in $\bigcup_{k=1}^{\infty} A_k$. Hence

$$\sum_{k=1}^{m}\sum_{j=1}^{n_k}|\nu(E_{j,k})| \leq |\nu|\Bigl(\bigcup_{k=1}^{\infty} A_k\Bigr).$$

Taking the supremum of the left side of the inequality above over all choices of $\{E_{j,k}\}$ satisfying 9.12 shows that

$$\sum_{k=1}^{m}|\nu|(A_k) \leq |\nu|\Bigl(\bigcup_{k=1}^{\infty} A_k\Bigr).$$

Because the inequality above holds for all $m \in \mathbf{Z}^+$, we have

$$\sum_{k=1}^{\infty}|\nu|(A_k) \leq |\nu|\Bigl(\bigcup_{k=1}^{\infty} A_k\Bigr).$$

To prove the inequality above in the other direction, suppose $E_1, \dots, E_n \in \mathcal{S}$ are disjoint sets such that $E_1 \cup \dots \cup E_n \subset \bigcup_{k=1}^{\infty} A_k$. Then

$$\begin{aligned}
\sum_{k=1}^{\infty} |\nu|(A_k) &\geq \sum_{k=1}^{\infty} \sum_{j=1}^{n} |\nu(E_j \cap A_k)| \\
&= \sum_{j=1}^{n} \sum_{k=1}^{\infty} |\nu(E_j \cap A_k)| \\
&\geq \sum_{j=1}^{n} \Big| \sum_{k=1}^{\infty} \nu(E_j \cap A_k) \Big| \\
&= \sum_{j=1}^{n} |\nu(E_j)|,
\end{aligned}$$

where the first line above follows from the definition of $|\nu|(A_k)$ and the last line above follows from the countable additivity of $\nu$.

The inequality above and the definition of $|\nu|\big(\bigcup_{k=1}^{\infty} A_k\big)$ imply that

$$\sum_{k=1}^{\infty} |\nu|(A_k) \geq |\nu|\Big(\bigcup_{k=1}^{\infty} A_k\Big),$$

completing the proof. ■

## The Banach Space of Measures

In this subsection, we will make the set of complex or real measures on a measurable space into a vector space and then into a Banach space.

**9.13 Definition** ***addition and scalar multiplication of measures***

Suppose $(X, \mathcal{S})$ is a measurable space. For complex measures $\nu, \mu$ on $(X, \mathcal{S})$ and $\alpha \in \mathbf{F}$, define complex measures $\nu + \mu$ and $\alpha\nu$ on $(X, \mathcal{S})$ by

$$(\nu + \mu)(E) = \nu(E) + \mu(E) \quad \text{and} \quad (\alpha\nu)(E) = \alpha\big(\nu(E)\big).$$

You should verify that if $\nu$, $\mu$, and $\alpha$ are as above, then $\nu + \mu$ and $\alpha\nu$ are complex measures on $(X, \mathcal{S})$. You should also verify that these natural definitions of addition and scalar multiplication make the set of complex (or real) measures on a measurable space $(X, \mathcal{S})$ into a vector space. We now introduce notation for this vector space.

**9.14 Definition** $\mathcal{M}_{\mathbf{F}}(\mathcal{S})$

Suppose $(X, \mathcal{S})$ is a measurable space. Then $\mathcal{M}_{\mathbf{F}}(\mathcal{S})$ denotes the vector space of real measures on $(X, \mathcal{S})$ if $\mathbf{F} = \mathbf{R}$ and denotes the vector space of complex measures on $(X, \mathcal{S})$ if $\mathbf{F} = \mathbf{C}$.

We use the terminology *total variation norm* below even though the object being defined is not obviously a norm (especially because it is not obvious that $\|\nu\| < \infty$ for every complex measure $\nu$). Soon we will justify this terminology.

**9.15 Definition** ***total variation norm of a complex measure***

Suppose $\nu$ is a complex measure on a measurable space $(X, \mathcal{S})$. The total variation norm of $\nu$, denoted $\|\nu\|$, is defined by

$$\|\nu\| = |\nu|(X).$$

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9.16 Example *total variation norm*

- If $\mu$ is a finite (positive) measure, then $\|\mu\| = \mu(X)$, as you should verify.
- If $\mu$ is a (positive) measure, $h \in \mathcal{L}^1(\mu)$, and $d\nu = h\, d\mu$, then $\|\nu\| = \|h\|_1$ (as follows from 9.10).

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The next result implies that if $\nu$ is a complex measure on a measurable space $(X, \mathcal{S})$, then $|\nu|(E) < \infty$ for every $E \in \mathcal{S}$.

9.17 ***Total variation norm is finite***

Suppose $(X, \mathcal{S})$ is a measurable space and $\nu \in \mathcal{M}_{\mathbf{F}}(\mathcal{S})$. Then $\|\nu\| < \infty$.

Proof First consider the case where $\mathbf{F} = \mathbf{R}$. Thus $\nu$ is a real measure on $(X, \mathcal{S})$. To begin this proof by contradiction, suppose $\|\nu\| = |\nu|(X) = \infty$.

We inductively choose a decreasing sequence $E_0 \supset E_1 \supset E_2 \supset \cdots$ of sets in $\mathcal{S}$ as follows: Start by choosing $E_0 = X$. Now suppose $n \geq 0$ and $E_n \in \mathcal{S}$ has been chosen with $|\nu|(E_n) = \infty$ and $|\nu(E_n)| \geq n$. Because $|\nu|(E_n) = \infty$, 9.9 implies that there exists $A \in \mathcal{S}$ such that $A \subset E_n$ and $|\nu(A)| \geq n + 1 + |\nu(E_n)|$, which implies that

$$|\nu(E_n \setminus A)| = |\nu(E_n) - \nu(A)| \geq |\nu(A)| - |\nu(E_n)| \geq n + 1.$$

Now

$$|\nu|(A) + |\nu|(E_n \setminus A) = |\nu|(E_n) = \infty$$

because the total variation measure $|\nu|$ is a (positive) measure (by 9.11). The equation above shows that at least one of $|\nu|(A)$ and $|\nu|(E_n \setminus A)$ is $\infty$. Let $E_{n+1} = A$ if $|\nu|(A) = \infty$ and let $E_{n+1} = E_n \setminus A$ if $|\nu|(A) < \infty$. Thus $E_n \supset E_{n+1}$, $|\nu|(E_{n+1}) = \infty$, and $|\nu(E_{n+1})| \geq n + 1$.

Now 9.7(d) implies that $\nu\bigl(\bigcap_{n=1}^{\infty} E_n\bigr) = \lim_{n\to\infty} \nu(E_n)$. However, $|\nu(E_n)| \geq n$ for each $n \in \mathbf{Z}^+$, and thus the limit in the last equation does not exist (in $\mathbf{R}$). This contradiction completes the proof in the case where $\nu$ is a real measure.

Consider now the case where $\mathbf{F} = \mathbf{C}$; thus $\nu$ is a complex measure on $(X, \mathcal{S})$. Then

$$|\nu|(X) \leq |\operatorname{Re} \nu|(X) + |\operatorname{Im} \nu|(X) < \infty,$$

where the last inequality follows from applying the real case to $\operatorname{Re} \nu$ and $\operatorname{Im} \nu$. ∎

The previous result tells us that if $(X, \mathcal{S})$ is a measurable space, then $\|\nu\| < \infty$ for all $\nu \in \mathcal{M}_{\mathbf{F}}(\mathcal{S})$. This implies (as the reader should verify) that the total variation norm $\|\cdot\|$ is a norm on $\mathcal{M}_{\mathbf{F}}(\mathcal{S})$. The next result shows that this norm makes $\mathcal{M}_{\mathbf{F}}(\mathcal{S})$ into a Banach space (in other words, every Cauchy sequence in this norm converges).

**9.18** ***The set of real or complex measures on $(X, \mathcal{S})$ is a Banach space***

Suppose $(X, \mathcal{S})$ is a measurable space. Then $\mathcal{M}_{\mathbf{F}}(\mathcal{S})$ is a Banach space with the total variation norm.

Proof Suppose $\nu_1, \nu_2, \ldots$ is a Cauchy sequence in $\mathcal{M}_{\mathbf{F}}(\mathcal{S})$. For each $E \in \mathcal{S}$, we have

$$\begin{aligned} |\nu_j(E) - \nu_k(E)| &= |(\nu_j - \nu_k)(E)| \\ &\leq |\nu_j - \nu_k|(E) \\ &\leq \|\nu_j - \nu_k\|. \end{aligned}$$

Thus $\nu_1(E), \nu_2(E), \ldots$ is a Cauchy sequence in $\mathbf{F}$ and hence converges. Thus we can define a function $\nu\colon \mathcal{S} \to \mathbf{F}$ by

$$\nu(E) = \lim_{j\to\infty} \nu_j(E).$$

To show that $\nu \in \mathcal{M}_{\mathbf{F}}(\mathcal{S})$, we must verify that $\nu$ is countably additive. To do this, suppose $E_1, E_2, \ldots$ is a disjoint sequence of sets in $\mathcal{S}$. Let $\varepsilon > 0$. Let $m \in \mathbf{Z}^+$ be such that

9.19 $$\|\nu_j - \nu_k\| \leq \varepsilon \quad \text{for all } j, k \geq m.$$

If $n \in \mathbf{Z}^+$ is such that

9.20 $$\sum_{k=n}^{\infty} |\nu_m(E_k)| \leq \varepsilon$$

[such an $n$ exists by applying 9.3(b) to $\nu_m$] and if $j \geq m$, then

$$\begin{aligned} \sum_{k=n}^{\infty} |\nu_j(E_k)| &\leq \sum_{k=n}^{\infty} |(\nu_j - \nu_m)(E_k)| + \sum_{k=n}^{\infty} |\nu_m(E_k)| \\ &\leq \sum_{k=n}^{\infty} |\nu_j - \nu_m|(E_k) + \varepsilon \\ &= |\nu_j - \nu_m|\Big(\bigcup_{k=n}^{\infty} E_k\Big) + \varepsilon \end{aligned}$$

9.21 $$\leq 2\varepsilon,$$

where the second line uses 9.20, the third line uses the countable additivity of the measure $|\nu_j - \nu_m|$ (see 9.11), and the fourth line uses 9.19.

If $\varepsilon$ and $n$ are as in the paragraph above, then

$$\begin{aligned}\Big|\nu\Big(\bigcup_{k=1}^{\infty} E_k\Big) - \sum_{k=1}^{n-1} \nu(E_k)\Big| &= \Big|\lim_{j\to\infty} \nu_j\Big(\bigcup_{k=1}^{\infty} E_k\Big) - \lim_{j\to\infty} \sum_{k=1}^{n-1} \nu_j(E_k)\Big| \\ &= \lim_{j\to\infty}\Big|\sum_{k=n}^{\infty} \nu_j(E_k)\Big| \\ &\le 2\varepsilon,\end{aligned}$$

where the second line uses the countable additivity of the measure $\nu_j$ and the third line uses 9.21. The inequality above implies that $\nu\big(\bigcup_{k=1}^{\infty} E_k\big) = \sum_{k=1}^{\infty} \nu(E_k)$, completing the proof that $\nu \in \mathcal{M}_{\mathbf{F}}(\mathcal{S})$.

We still need to prove that $\lim_{k\to\infty}\|\nu - \nu_k\| = 0$. To do this, suppose $\varepsilon > 0$. Let $m \in \mathbf{Z}^+$ be such that

9.22 $$\|\nu_j - \nu_k\| \le \varepsilon \quad \text{for all } j, k \ge m.$$

Suppose $k \ge m$. Suppose also that $E_1, \dots, E_n \in \mathcal{S}$ are disjoint subsets of $X$. Then

$$\sum_{\ell=1}^{n} |(\nu - \nu_k)(E_\ell)| = \lim_{j\to\infty} \sum_{\ell=1}^{n} |(\nu_j - \nu_k)(E_\ell)| \le \varepsilon,$$

where the last inequality follows from 9.22 and the definition of the total variation norm. The inequality above implies that $\|\nu - \nu_k\| \le \varepsilon$, completing the proof. ∎

## EXERCISES 9A

**1** Prove or give a counterexample: If $\nu$ is a real measure on a measurable space $(X, \mathcal{S})$ and $A, B \in \mathcal{S}$ are such that $\nu(A) \ge 0$ and $\nu(B) \ge 0$, then $\nu(A \cup B) \ge 0$.

**2** Suppose $\nu$ is a real measure on $(X, \mathcal{S})$. Define $\mu\colon \mathcal{S} \to [0, \infty)$ by

$$\mu(E) = |\nu(E)|.$$

Prove that $\mu$ is a (positive) measure on $(X, \mathcal{S})$ if and only if the range of $\nu$ is contained in $[0, \infty)$ or the range of $\nu$ is contained in $(-\infty, 0]$.

**3** Suppose $\nu$ is a complex measure on a measurable space $(X, \mathcal{S})$. Prove that $|\nu|(X) = \nu(X)$ if and only if $\nu$ is a (positive) measure.

**4** Suppose $\nu$ is a complex measure on a measurable space $(X, \mathcal{S})$. Prove that if $E \in \mathcal{S}$ then

$$|\nu|(E) = \sup\Big\{\sum_{k=1}^{\infty} |\nu(E_k)| : E_1, E_2, \dots \text{ is a disjoint sequence in } \mathcal{S} \text{ such that } E = \bigcup_{k=1}^{\infty} E_k\Big\}.$$

**5** Suppose $\mu$ is a (positive) measure on a measurable space $(X, \mathcal{S})$ and $h$ is a nonnegative function in $\mathcal{L}^1(\mu)$. Let $\nu$ be the (positive) measure on $(X, \mathcal{S})$ defined by $d\nu = h\,d\mu$. Prove that

$$\int f\,d\nu = \int fh\,d\mu$$

for all $\mathcal{S}$-measurable functions $f\colon X \to [0, \infty]$.

**6** Suppose $(X, \mathcal{S}, \mu)$ is a (positive) measure space. Prove that

$$\{h\,d\mu : h \in \mathcal{L}^1(\mu)\}$$

is a closed subspace of $\mathcal{M}_{\mathbf{F}}(\mathcal{S})$.

**7** (a) Suppose $\mathcal{B}$ is the collection of Borel subsets of $\mathbf{R}$. Show that the Banach space $\mathcal{M}_{\mathbf{F}}(\mathcal{B})$ is not separable.

(b) Give an example of a measurable space $(X, \mathcal{S})$ such that the Banach space $\mathcal{M}_{\mathbf{F}}(\mathcal{S})$ is infinite dimensional and separable.

**8** Suppose $t > 0$ and $\lambda$ is Lebesgue measure on the $\sigma$-algebra of Borel subsets of $[0, t]$. Suppose $h\colon [0, t] \to \mathbf{C}$ is the function defined by

$$h(x) = \cos x + i \sin x.$$

Let $\nu$ be the complex measure defined by $d\nu = h\,d\lambda$.

(a) Show that $\|\nu\| = t$.

(b) Show that if $E_1, E_2, \dots$ is a sequence of disjoint Borel subsets of $[0, t]$, then

$$\sum_{k=1}^{\infty} |\nu(E_k)| < t.$$

[*This exercise shows that the supremum in the definition of* $|\nu|([0, t])$ *is not attained, even if countably many disjoint sets are allowed.*]

**9** Give an example to show that 9.9 can fail if the hypothesis that $\nu$ is a real measure is replaced by the hypothesis that $\nu$ is a complex measure.

**10** Suppose $(X, \mathcal{S})$ is a measurable space with $\mathcal{S} \neq \{\emptyset, X\}$. Prove that the total variation norm on $\mathcal{M}_{\mathbf{F}}(\mathcal{S})$ does not come from an inner product. In other words, show that there does not exist an inner product $\langle \cdot, \cdot \rangle$ on $\mathcal{M}_{\mathbf{F}}(\mathcal{S})$ such that $\|\nu\| = \langle \nu, \nu \rangle^{1/2}$ for all $\nu \in \mathcal{M}_{\mathbf{F}}(\mathcal{S})$, where $\|\cdot\|$ is the usual total variation norm on $\mathcal{M}_{\mathbf{F}}(\mathcal{S})$.

**11** For $(X, \mathcal{S})$ a measurable space and $b \in X$, define a finite (positive) measure $\delta_b$ on $(X, \mathcal{S})$ by

$$\delta_b(E) = \begin{cases} 1 & \text{if } b \in E, \\ 0 & \text{if } b \notin E \end{cases}$$

for $E \in \mathcal{S}$.

(a) Show that if $b, c \in X$, then $\|\delta_b + \delta_c\| = 2$.

(b) Give an example of a measurable space $(X, \mathcal{S})$ and $b, c \in X$ with $b \neq c$ such that $\|\delta_b - \delta_c\| \neq 2$.

# 9B Decomposition Theorems

## Hahn Decomposition Theorem

The next result shows that a real measure on a measurable space $(X, \mathcal{S})$ decomposes $X$ into two disjoint measurable sets, on one of which all subsets have nonnegative measure and on the other of which all subsets have nonpositive measure.

The decomposition in the result below is not unique because a subset $D$ of $X$ with $|\nu|(D) = 0$ could be shifted from $A$ to $B$ or from $B$ to $A$. However, Exercise 1 at the end of this section shows that the Hahn decomposition is almost unique.

**9.23 *Hahn Decomposition Theorem***

Suppose $\nu$ is a real measure on a measurable space $(X, \mathcal{S})$. Then there exist sets $A, B \in \mathcal{S}$ such that

- $A \cup B = X$ and $A \cap B = \varnothing$;
- $\nu(E) \geq 0$ for every $E \in \mathcal{S}$ with $E \subset A$;
- $\nu(E) \leq 0$ for every $E \in \mathcal{S}$ with $E \subset B$.

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**9.24 Example** *Hahn Decomposition*

Suppose $\mu$ is a (positive) measure on a measurable space $(X, \mathcal{S})$, $h \in \mathcal{L}^1(\mu)$ is real valued, and $d\nu = h\, d\mu$. Then a Hahn decomposition of the real measure $\nu$ is obtained by setting

$$A = \{x \in X : h(x) \geq 0\} \quad \text{and} \quad B = \{x \in X : h(x) < 0\}.$$

---

**Proof of 9.23** Let

$$a = \sup\{\nu(E) : E \in \mathcal{S}\}.$$

Thus $a \leq \|\nu\| < \infty$, where the last inequality comes from 9.17. For each $j \in \mathbf{Z}^+$, let $A_j \in \mathcal{S}$ be such that

9.25
$$\nu(A_j) \geq a - \frac{1}{2^j}.$$

Temporarily fix $k \in \mathbf{Z}^+$. We will show by induction on $n$ that if $n \in \mathbf{Z}^+$ with $n \geq k$, then

9.26
$$\nu\Big(\bigcup_{j=k}^{n} A_j\Big) \geq a - \sum_{j=k}^{n} \frac{1}{2^j}.$$

To get started with the induction, note that if $n = k$ then 9.26 holds because in this case 9.26 becomes 9.25. Now for the induction step, assume that $n \geq k$ and that 9.26 holds. Then

$$\begin{aligned}
\nu\Big(\bigcup_{j=k}^{n+1} A_j\Big) &= \nu\Big(\bigcup_{j=k}^{n} A_j\Big) + \nu(A_{n+1}) - \nu\Big(\big(\bigcup_{j=k}^{n} A_j\big) \cap A_{n+1}\Big)\\
&\geq \Big(a - \sum_{j=k}^{n} \frac{1}{2^j}\Big) + \Big(a - \frac{1}{2^{n+1}}\Big) - a\\
&= a - \sum_{j=k}^{n+1} \frac{1}{2^j},
\end{aligned}$$

where the first line follows from 9.7(b) and the second line follows from 9.25 and 9.26. We have now verified that 9.26 holds if $n$ is replaced by $n+1$, completing the proof by induction of 9.26.

The sequence of sets $A_k, A_k \cup A_{k+1}, A_k \cup A_{k+1} \cup A_{k+2}, \dots$ is increasing. Thus taking the limit as $n \to \infty$ of both sides of 9.26 and using 9.7(c) gives

9.27
$$\nu\Big(\bigcup_{j=k}^{\infty} A_j\Big) \geq a - \frac{1}{2^{k-1}}.$$

Now let
$$A = \bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} A_j.$$

The sequence of sets $\bigcup_{j=1}^{\infty} A_j, \bigcup_{j=2}^{\infty} A_j, \dots$ is decreasing. Thus 9.27 and 9.7(d) imply that $\nu(A) \geq a$. The definition of $a$ now implies that

$$\nu(A) = a.$$

Suppose $E \in \mathcal{S}$ and $E \subset A$. Then $\nu(A) = a \geq \nu(A \setminus E)$. Thus we have $\nu(E) = \nu(A) - \nu(A \setminus E) \geq 0$, which proves the second bullet point of this result.

Let $B = X \setminus A$. Suppose $E \in \mathcal{S}$ and $E \subset B$. Then $\nu(A \cup E) \leq a = \nu(A)$. Thus $\nu(E) = \nu(A \cup E) - \nu(A) \leq 0$, which proves the third bullet point of this result. ∎

## Jordan Decomposition Theorem

You should think of two complex or positive measures on a measurable space $(X, \mathcal{S})$ as being singular with respect to each other if the two measures live on different sets. Here is the formal definition.

**9.28 Definition** ***singular measures***

Suppose $\nu$ and $\mu$ are complex or positive measures on a measurable space $(X, \mathcal{S})$. Then $\nu$ and $\mu$ are called *singular* (with respect to each other), denoted $\nu \perp \mu$, if there exist sets $A, B \in \mathcal{S}$ such that

- $A \cup B = X$ and $A \cap B = \varnothing$;
- $\nu(E) = \nu(E \cap A)$ and $\mu(E) = \mu(E \cap B)$ for all $E \in \mathcal{S}$.

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9.29 **Example** *singular measures*

Suppose $\lambda$ is Lebesgue measure on the $\sigma$-algebra $\mathcal{B}$ of Borel subsets of $\mathbf{R}$.

- Define positive measures $\nu, \mu$ on $(\mathbf{R}, \mathcal{B})$ by

$$\nu(E) = |E \cap (-\infty, 0)| \quad \text{and} \quad \mu(E) = |E \cap (2, 3)|$$

for $E \in \mathcal{B}$. Then $\nu \perp \mu$ because $\nu$ lives on $(-\infty, 0)$ and $\mu$ lives on $[0, \infty)$. Neither $\nu$ nor $\mu$ is singular with respect to $\lambda$.

- Let $r_1, r_2, \ldots$ be a list of the rational numbers. Suppose $w_1, w_2, \ldots$ is a bounded sequence of complex numbers. Define a complex measure $\nu$ on $(\mathbf{R}, \mathcal{B})$ by

$$\nu(E) = \sum_{\{k \in \mathbf{Z}^+ : r_k \in E\}} \frac{w_k}{2^k}$$

for $E \in \mathcal{B}$. Then $\nu \perp \lambda$ because $\nu$ lives on $\mathbf{Q}$ and $\lambda$ lives on $\mathbf{R} \setminus \mathbf{Q}$.

---

The hard work for proving the next result has already been done in proving the Hahn Decomposition Theorem (9.23).

**9.30** ***Jordan Decomposition Theorem***

- Every real measure is the difference of two finite (positive) measures that are singular with respect to each other.
- More precisely, suppose $\nu$ is a real measure on a measurable space $(X, \mathcal{S})$. Then there exist unique finite (positive) measures $\nu^+$ and $\nu^-$ on $(X, \mathcal{S})$ such that

9.31
$$\nu = \nu^+ - \nu^- \quad \text{and} \quad \nu^+ \perp \nu^-.$$

Furthermore,

$$|\nu| = \nu^+ + \nu^-.$$

**Proof** Let $X = A \cup B$ be a Hahn decomposition of $\nu$ as in 9.23. Define functions $\nu^+ : \mathcal{S} \to [0, \infty)$ and $\nu^- : \mathcal{S} \to [0, \infty)$ by

$$\nu^+(E) = \nu(E \cap A) \quad \text{and} \quad \nu^-(E) = -\nu(E \cap B).$$

The countable additivity of $\nu$ implies $\nu^+$ and $\nu^-$ are finite (positive) measures on $(X, \mathcal{S})$, with $\nu = \nu^+ - \nu^-$ and $\nu^+ \perp \nu^-$.

*Camille Jordan (1838–1922) is also known for matrices that are 0 except along the diagonal and the line above it.*

The definition of the total variation measure and 9.31 imply that $|\nu| = \nu^+ + \nu^-$, as you should verify.

The equations $\nu = \nu^+ - \nu^-$ and $|\nu| = \nu^+ + \nu^-$ imply that

$$\nu^+ = \frac{|\nu| + \nu}{2} \quad \text{and} \quad \nu- = \frac{|\nu| - \nu}{2}.$$

Thus the finite (positive) measures $\nu^+$ and $\nu^-$ are uniquely determined by $\nu$ and the conditions in 9.31. ∎

 

## Lebesgue Decomposition Theorem

The next definition captures the notion of one measure having more sets of measure 0 than another measure.

**9.32 Definition** ***absolutely continuous;*** $\ll$

Suppose $\nu$ is a complex measure on a measurable space $(X, \mathcal{S})$ and $\mu$ is a (positive) measure on $(X, \mathcal{S})$. Then $\nu$ is called *absolutely continuous* with respect to $\mu$, denoted $\nu \ll \mu$, if

$$\nu(E) = 0 \text{ for every set } E \in \mathcal{S} \text{ with } \mu(E) = 0.$$

9.33 Example *absolute continuity*

The reader should verify all the following examples:

- If $\mu$ is a (positive) measure and $h \in \mathcal{L}^1(\mu)$, then $h\,d\mu \ll \mu$.
- If $\nu$ is a real measure, then $\nu^+ \ll |\nu|$ and $\nu^- \ll |\nu|$.
- If $\nu$ is a complex measure, then $\nu \ll |\nu|$.
- If $\nu$ is a complex measure, then $\operatorname{Re}\nu \ll |\nu|$ and $\operatorname{Im}\nu \ll |\nu|$.
- Every measure on a measurable space $(X, \mathcal{S})$ is absolutely continuous with respect to counting measure on $(X, \mathcal{S})$.

The next result should help you think that absolute continuity and singularity are two extreme possibilities for the relationship between two complex measures.

**9.34** ***Absolutely continuous and singular implies*** 0

Suppose $\mu$ is a (positive) measure on a measurable space $(X, \mathcal{S})$. Then the only complex measure on $(X, \mathcal{S})$ that is both absolutely continuous and singular with respect to $\mu$ is the 0 measure.

Proof Suppose $\nu$ is a complex measure on $(X, \mathcal{S})$ such that $\nu \ll \mu$ and $\nu \perp \mu$. Thus there exist sets $A, B \in \mathcal{S}$ such that $A \cup B = X$, $A \cap B = \varnothing$, and $\nu(E) = \nu(E \cap A)$ and $\mu(E) = \mu(E \cap B)$ for every $E \in \mathcal{S}$.

Suppose $E \in \mathcal{S}$. Then

$$\mu(E \cap A) = \mu\big((E \cap A) \cap B\big) = \mu(\varnothing) = 0.$$

Because $\nu \ll \mu$, this implies that $\nu(E \cap A) = 0$. Thus $\nu(E) = 0$. Hence $\nu$ is the 0 measure. ■

Our next result states that a (positive) measure on a measurable space $(X, \mathcal{S})$ determines a decomposition of each complex measure on $(X, \mathcal{S})$ as the sum of the two extreme types of complex measures (absolute continuity and singularity).

**9.35 *Lebesgue Decomposition Theorem***

Suppose $\mu$ is a (positive) measure on a measurable space $(X, \mathcal{S})$.

- Every complex measure on $(X, \mathcal{S})$ is the sum of a complex measure absolutely continuous with respect to $\mu$ and a complex measure singular with respect to $\mu$.
- More precisely, suppose $\nu$ is a complex measure on $(X, \mathcal{S})$. Then there exist unique complex measures $\nu_a$ and $\nu_s$ on $(X, \mathcal{S})$ such that $\nu = \nu_a + \nu_s$ and

$$\nu_a \ll \mu \quad \text{and} \quad \nu_s \perp \mu.$$

**Proof** Let

$$b = \sup\{|\nu|(B) : B \in \mathcal{S} \text{ and } \mu(B) = 0\}.$$

For each $k \in \mathbf{Z}^+$, let $B_k \in \mathcal{S}$ be such that

$$|\nu|(B_k) \geq b - \tfrac{1}{k} \quad \text{and} \quad \mu(B_k) = 0.$$

Let

$$B = \bigcup_{k=1}^{\infty} B_k.$$

Then $\mu(B) = 0$ and $|\nu|(B) = b$.

Let $A = X \setminus B$. Define complex measures $\nu_a$ and $\nu_s$ on $(X, \mathcal{S})$ by

$$\nu_a(E) = \nu(E \cap A) \quad \text{and} \quad \nu_s(E) = \nu(E \cap B).$$

Clearly $\nu = \nu_a + \nu_s$.

If $E \in \mathcal{S}$, then

$$\mu(E) = \mu(E \cap A) + \mu(E \cap B) = \mu(E \cap A),$$

where the last equality holds because $\mu(B) = 0$. The equation above implies that $\nu_s \perp \mu$.

To prove that $\nu_a \ll \mu$, suppose $E \in \mathcal{S}$ and $\mu(E) = 0$. Then $\mu(B \cup E) = 0$ and hence

$$b \geq |\nu|(B \cup E) = |\nu|(B) + |\nu|(E \setminus B) = b + |\nu|(E \setminus B),$$

which implies that $|\nu|(E \setminus B) = 0$. Thus

*The construction of $\nu_a$ and $\nu_s$ shows that if $\nu$ is a positive (or real) measure, then so are $\nu_a$ and $\nu_s$.*

$$\nu_a(E) = \nu(E \cap A) = \nu(E \setminus B) = 0,$$

which implies that $\nu_a \ll \mu$.

We have now proved all parts of this result except the uniqueness of the Lebesgue decomposition. To prove the uniqueness, suppose $\nu_1$ and $\nu_2$ are complex measures on $(X, \mathcal{S})$ such that $\nu_1 \ll \mu$, $\nu_2 \perp \mu$, and $\nu = \nu_1 + \nu_2$. Then

$$\nu_1 - \nu_a = \nu_s - \nu_2.$$

The left side of the equation above is absolutely continuous with respect to $\mu$ and the right side is singular with respect to $\mu$. Thus both sides are both absolutely continuous and singular with respect to $\mu$. Thus 9.34 implies that $\nu_1 = \nu_a$ and $\nu_2 = \nu_s$. ∎

## Radon–Nikodym Theorem

If $\mu$ is a (positive) measure, $h \in \mathcal{L}^1(\mu)$, and $d\nu = h\,d\mu$, then $\nu \ll \mu$. The next result gives the important converse—if $\mu$ is $\sigma$-finite, then every complex measure that is absolutely continuous with respect to $\mu$ is of the form $h\,d\mu$ for some $h \in \mathcal{L}^1(\mu)$. The hypothesis that $\mu$ is $\sigma$-finite cannot be deleted.

*The result below was first proved by Radon and Otto Nikodym (1887–1974).*

**9.36 *Radon–Nikodym Theorem***

Suppose $\mu$ is a (positive) $\sigma$-finite measure on a measurable space $(X, \mathcal{S})$. Suppose $\nu$ is a complex measure on $(X, \mathcal{S})$ such that $\nu \ll \mu$. Then there exists $h \in \mathcal{L}^1(\mu)$ such that $d\nu = h\,d\mu$.

**Proof** First consider the case where both $\mu$ and $\nu$ are finite (positive) measures. Define $\varphi\colon L^2(\nu+\mu) \to \mathbf{R}$ by

9.37 $$\varphi(f) = \int f\,d\nu.$$

To show that $\varphi$ is well defined, first note that if $f \in \mathcal{L}^2(\nu+\mu)$, then

9.38 $$\int |f|\,d\nu \le \int |f|\,d(\nu+\mu) \le \bigl(\nu(X)+\mu(X)\bigr)^{1/2}\|f\|_{L^2(\nu+\mu)} < \infty,$$

where the middle inequality follows from Hölder's Inequality (7.9) applied to the functions 1 and $f$. The inequality above shows that $\int f\,d\nu$ makes sense for $f \in \mathcal{L}^2(\nu+\mu)$. Furthermore, if two functions in $\mathcal{L}^2(\nu+\mu)$ differ only on a set of $(\nu+\mu)$-measure 0, then they differ only on a set of $\nu$-measure 0. Thus $\varphi$ as defined in 9.37 makes sense as a linear functional on $L^2(\nu+\mu)$.

Because $|\varphi(f)| \le \int |f|\,d\nu$, 9.38 shows that $\varphi$ is a bounded linear functional on $L^2(\nu+\mu)$. The Riesz Representation Theorem (8.47) now implies that there exists $g \in \mathcal{L}^2(\nu+\mu)$ such that

*The clever idea of using Hilbert space techniques in this proof comes from John von Neumann (1903–1957).*

$$\int f\,d\nu = \int fg\,d(\nu+\mu)$$

for all $f \in \mathcal{L}^2(\nu+\mu)$. Hence

9.39 $$\int f(1-g)\,d\nu = \int fg\,d\mu$$

for all $f \in \mathcal{L}^2(\nu+\mu)$.

If $f$ equals the characteristic function of $\{x \in X : g(x) \ge 1\}$, then the left side of 9.39 is less than or equal to 0 and the right side of 9.39 is greater than or equal to 0; thus both sides are 0. Setting the right side of 9.39 equal to 0 implies (with this choice of $f$) that $\mu(\{x \in X : g(x) \ge 1\}) = 0$.

Similarly, if $f$ equals the characteristic function of $\{x \in X : g(x) < 0\}$, then the left side of 9.39 is greater than or equal to 0 and the right side of 9.39 is less than or equal to 0; thus both sides are 0. Setting the right side of 9.39 equal to 0 implies (with this choice of $f$) that $\mu(\{x \in X : g(x) < 0\}) = 0$.

Because $\nu \ll \mu$, the two previous paragraphs imply that

$$\nu(\{x \in X : g(x) \geq 1\}) = 0 \quad \text{and} \quad \nu(\{x \in X : g(x) < 0\}) = 0.$$

Thus we can modify $g$ (for example by redefining $g$ to be $\frac{1}{2}$ on the two sets appearing above; both those sets have $\nu$-measure 0 and $\mu$-measure 0) and from now on we can assume that $0 \leq g(x) < 1$ for all $x \in X$ and that 9.39 holds for all $f \in \mathcal{L}^2(\nu + \mu)$. Hence we can define $h\colon X \to [0, \infty)$ by

$$h(x) = \frac{g(x)}{1 - g(x)}.$$

*Taking $f = \chi_E/(1-g)$ in 9.39 would give $\nu(E) = \int_E h\, d\mu$, but this function $f$ might not be in $\mathcal{L}^2(\nu + \mu)$ and thus we need to be a bit more careful.*

Suppose $E \in \mathcal{S}$. For each $k \in \mathbf{Z}^+$, let

$$f_k(x) = \begin{cases} \frac{\chi_E(x)}{1-g(x)} & \text{if } \frac{\chi_E(x)}{1-g(x)} \leq k, \\ 0 & \text{otherwise.} \end{cases}$$

Then $f_k \in \mathcal{L}^2(\nu + \mu)$. Now 9.39 implies

$$\int f_k(1-g)\, d\nu = \int f_k g\, d\mu.$$

Taking the limit as $k \to \infty$ and using the Monotone Convergence Theorem (3.11) shows that

9.40 $$\int_E 1\, d\nu = \int_E h\, d\mu.$$

Thus $d\nu = h\, d\mu$, completing the proof in the case where both $\nu$ and $\mu$ are (positive) finite measures [note that $h \in \mathcal{L}^1(\mu)$ because $h$ is a nonnegative function and we can take $E = X$ in the equation above].

Now relax the assumption on $\mu$ to the hypothesis that $\mu$ is a $\sigma$-finite measure. Thus there exists an increasing sequence $X_1 \subset X_2 \subset \cdots$ of sets in $\mathcal{S}$ such that $\bigcup_{k=1}^{\infty} X_k = X$ and $\mu(X_k) < \infty$ for each $k \in \mathbf{Z}^+$. For $k \in \mathbf{Z}^+$, let $\nu_k$ and $\mu_k$ denote the restrictions of $\nu$ and $\mu$ to the $\sigma$-algebra on $X_k$ consisting of those sets in $\mathcal{S}$ that are subsets of $X_k$. Then $\nu_k \ll \mu_k$. Thus by the case we have already proved, there exists a nonnegative function $h_k \in \mathcal{L}^1(\mu_k)$ such that $d\nu_k = h_k\, d\mu_k$. If $j < k$, then

$$\int_E h_j\, d\mu = \nu(E) = \int_E h_k\, d\mu$$

for every set $E \in \mathcal{S}$ with $E \subset X_j$; thus $\mu(\{x \in X_j : h_j(x) \neq h_k(x)\}) = 0$. Hence there exists an $\mathcal{S}$-measurable function $h\colon X \to [0, \infty)$ such that

$$\mu(\{x \in X_k : h(x) \neq h_k(x)\}) = 0$$

for every $k \in \mathbf{Z}^+$. The Monotone Convergence Theorem (3.11) can now be used to show that 9.40 holds for every $E \in \mathcal{S}$. Thus $d\nu = h\, d\mu$, completing the proof in the case where $\nu$ is a (positive) finite measure.

Now relax the assumption on $\nu$ to the assumption that $\nu$ is a real measure. The measure $\nu$ equals one-half the difference of the two positive (finite) measures $|\nu| + \nu$ and $|\nu| - \nu$, each of which is absolutely continuous with respect to $\mu$. By the case proved in the previous paragraph, there exist $h_+, h_- \in \mathcal{L}^1(\mu)$ such that

$$d(|\nu| + \nu) = h_+ \, d\mu \quad \text{and} \quad d(|\nu| - \nu) = h_- \, d\mu.$$

Taking $h = \frac{1}{2}(h_+ - h_-)$, we have $d\nu = h\, d\mu$, completing the proof in the case where $\nu$ is a real measure.

Finally, if $\nu$ is a complex measure, apply the result in the previous paragraph to the real measures $\operatorname{Re} \nu$, $\operatorname{Im} \nu$, producing $h_{\mathrm{Re}}, h_{\mathrm{Im}} \in \mathcal{L}^1(\mu)$ such that $d(\operatorname{Re} \nu) = h_{\mathrm{Re}}\, d\mu$ and $d(\operatorname{Im} \nu) = h_{\mathrm{Im}}\, d\mu$. Taking $h = h_{\mathrm{Re}} + i h_{\mathrm{Im}}$, we have $d\nu = h\, d\mu$, completing the proof in the case where $\nu$ is a complex measure. ■

The function $h$ provided by the Radon–Nikodym Theorem is unique up to changes on sets with $\mu$-measure 0. If we think of $h$ as an element of $L^1(\mu)$ instead of $\mathcal{L}^1(\mu)$, then the choice of $h$ is unique.

When $d\nu = h\, d\mu$, the notation $h = \frac{d\nu}{d\mu}$ is used by some authors, and $h$ is called the *Radon–Nikodym derivative* of $\nu$ with respect to $\mu$.

The next result is a nice consequence of the Radon–Nikodym Theorem.

**9.41** ***If $\nu$ is a complex measure, then $d\nu = h\, d|\nu|$ for some $h$ with $|h(x)| = 1$***

(a) Suppose $\nu$ is a real measure on a measurable space $(X, \mathcal{S})$. Then there exists an $\mathcal{S}$-measurable function $h\colon X \to \{-1, 1\}$ such that $d\nu = h\, d|\nu|$.

(b) Suppose $\nu$ is a complex measure on a measurable space $(X, \mathcal{S})$. Then there exists an $\mathcal{S}$-measurable function $h\colon X \to \{z \in \mathbf{C} : |z| = 1\}$ such that $d\nu = h\, d|\nu|$.

Proof Because $\nu \ll |\nu|$, the Radon–Nikodym Theorem (9.36) tells us that there exists $h \in \mathcal{L}^1(|\nu|)$ (with $h$ real valued if $\nu$ is a real measure) such that $d\nu = h\, d|\nu|$. Now 9.10 implies that $d|\nu| = |h|\, d|\nu|$, which implies that $|h| = 1$ almost everywhere (with respect to $|\nu|$). Refine $h$ to be 1 on the set $\{x \in X : |h(x)| \neq 1\}$, which gives the desired result. ■

We could have proved part (a) of the result above by taking $h = \chi_A - \chi_B$ in the Hahn Decomposition Theorem (9.23).

Conversely, we could give a new proof of Hahn Decomposition Theorem by using part (a) of the result above and taking

$$A = \{x \in X : h(x) = 1\} \quad \text{and} \quad B = \{x \in X : h(x) = -1\}.$$

We could also give a new proof of the Jordan Decomposition Theorem (9.30) by using part (a) of the result above and taking

$$\nu^+ = \chi_{\{x \in X : h(x) = 1\}}\, d|\nu| \quad \text{and} \quad \nu^- = \chi_{\{x \in X : h(x) = -1\}}\, d|\nu|.$$

## Dual Space of $L^p(\mu)$

Recall that the dual space of a normed vector space $V$ is the Banach space of bounded linear functionals on $V$; the dual space of $V$ is denoted by $V'$. Recall also that if $1 \le p \le \infty$, then the dual exponent $p'$ is defined by the equation $\frac{1}{p} + \frac{1}{p'} = 1$.

The dual space of $\ell^p$ can be identified with $\ell^{p'}$ for $1 \le p < \infty$, as we saw in 7.25. We are now ready to prove the analogous result for an arbitrary (positive) measure, identifying the dual space of $L^p(\mu)$ with $L^{p'}(\mu)$ [with the mild restriction that $\mu$ is $\sigma$-finite if $p = 1$]. In the special case where $\mu$ is counting measure on $\mathbf{Z}^+$, this new result reduces to the previous result about $\ell^p$.

For $1 < p < \infty$, the next result differs from 7.24 by only one word, with "to" in 7.24 changed to "onto" below. Thus we already know (and will use in the proof) that the map $h \mapsto \varphi_h$ is a one-to-one linear map from $L^{p'}(\mu)$ to $\big(L^p(\mu)\big)'$ and that $\|\varphi_h\| = \|h\|_{p'}$ for all $h \in L^{p'}(\mu)$. The new aspect of the result below is the assertion that every bounded linear functional on $L^p(\mu)$ is of the form $\varphi_h$ for some $h \in L^{p'}(\mu)$. The key tool that we will use in proving this new assertion is the Radon–Nikodym Theorem.

**9.42 *Dual space of $L^p(\mu)$ is $L^{p'}(\mu)$***

Suppose $\mu$ is a (positive) measure and $1 \le p < \infty$ [with the additional hypothesis that $\mu$ is a $\sigma$-finite measure if $p = 1$]. For $h \in L^{p'}(\mu)$, define $\varphi_h \colon L^p(\mu) \to \mathbf{F}$ by

$$\varphi_h(f) = \int fh \, d\mu.$$

Then $h \mapsto \varphi_h$ is a one-to-one linear map from $L^{p'}(\mu)$ onto $\big(L^p(\mu)\big)'$. Furthermore, $\|\varphi_h\| = \|h\|_{p'}$ for all $h \in L^{p'}(\mu)$.

**Proof** The case $p = 1$ will be left to the reader as an exercise. Thus assume $1 < p < \infty$.

Suppose $\mu$ is a (positive) measure on a measurable space $(X, \mathcal{S})$ and $\varphi$ is a bounded linear functional on $L^p(\mu)$; in other words, suppose $\varphi \in \big(L^p(\mu)\big)'$.

Consider first the case where $\mu$ is a finite (positive) measure. Define a function $\nu \colon \mathcal{S} \to [0, \infty)$ by

$$\nu(E) = \varphi(\chi_E).$$

If $E_1, E_2, \dots$ are disjoint sets in $\mathcal{S}$, then

$$\nu\Big(\bigcup_{k=1}^{\infty} E_k\Big) = \varphi\Big(\chi_{\bigcup_{k=1}^{\infty} E_k}\Big) = \varphi\Big(\sum_{k=1}^{\infty} \chi_{E_k}\Big) = \sum_{k=1}^{\infty} \varphi(\chi_{E_k}) = \sum_{k=1}^{\infty} \nu(E_k),$$

where the infinite sum in the third term converges in the $L^p(\mu)$-norm to $\chi_{\bigcup_{k=1}^{\infty} E_k}$, and the third equality holds because $\varphi$ is a continuous linear functional. The equation above shows that $\nu$ is countably additive. Thus $\nu$ is a complex measure on $(X, \mathcal{S})$ [and is a real measure if $\mathbf{F} = \mathbf{R}$].

If $E \in \mathcal{S}$ and $\mu(E) = 0$, then $\chi_E$ is the 0 element of $L^p(\mu)$, which implies that $\varphi(\chi_E) = 0$, which means that $\nu(E) = 0$. Hence $\nu \ll \mu$. By the Radon–Nikodym Theorem (9.36), there exists $h \in \mathcal{L}^1(\mu)$ such that $d\nu = h\,d\mu$. Hence

$$\varphi(\chi_E) = \nu(E) = \int_E h\,d\mu = \int \chi_E h\,d\mu$$

for every $E \in \mathcal{S}$. The equation above, along with the linearity of $\varphi$, implies that

9.43 $$\varphi(f) = \int fh\,d\mu \quad \text{for every simple } \mathcal{S}\text{-measurable function } f\colon X \to \mathbf{F}.$$

Because every bounded $\mathcal{S}$-measurable function is the uniform limit on $X$ of a sequence of simple $\mathcal{S}$-measurable functions (see 2.82), we can conclude from 9.43 that

9.44 $$\varphi(f) = \int fh\,d\mu \quad \text{for every } f \in L^\infty(\mu).$$

For $k \in \mathbf{Z}^+$, let

$$E_k = \{x \in X : 0 < |h(x)| \le k\}$$

and define $f_k \in L^p(\mu)$ by

9.45 $$f_k(x) = \begin{cases} \overline{h(x)}\,|h(x)|^{p'-2} & \text{if } x \in E_k, \\ 0 & \text{otherwise.} \end{cases}$$

Now

$$\int |h|^{p'} \chi_{E_k}\,d\mu = \varphi(f_k) \le \|\varphi\|\,\|f_k\|_p = \|\varphi\| \Big(\int |h|^{p'} \chi_{E_k}\,d\mu\Big)^{1/p},$$

where the first equality follows from 9.44 and 9.45, and the last equality follows from 9.45 [which implies that $|f_k(x)|^p = |h(x)|^{p'} \chi_{E_k}(x)$ for $x \in X$]. After dividing by $\big(\int |h|^{p'} \chi_{E_k}\,d\mu\big)^{1/p}$, the inequality between the first and last terms above becomes

$$\|h\chi_{E_k}\|_{p'} \le \|\varphi\|.$$

Taking the limit as $k \to \infty$ shows, via the Monotone Convergence Theorem (3.11), that

$$\|h\|_{p'} \le \|\varphi\|.$$

Thus $h \in L^{p'}(\mu)$. Because each $f \in L^p(\mu)$ can be approximated in the $L^p(\mu)$ norm by functions in $L^\infty(\mu)$, 9.44 now shows that $\varphi = \varphi_h$, completing the proof in the case where $\mu$ is a finite (positive) measure.

Now relax the assumption that $\mu$ is a finite (positive) measure to the hypothesis that $\mu$ is a (positive) measure. For $E \in \mathcal{S}$, let $\mathcal{S}_E = \{A \in \mathcal{S} : A \subset E\}$ and let $\mu_E$ be the (positive) measure on $(E, \mathcal{S}_E)$ defined by $\mu_E(A) = \mu(A)$ for $A \in \mathcal{S}_E$. We can identify $L^p(\mu_E)$ with the subspace of functions in $L^p(\mu)$ that vanish (almost everywhere) outside $E$. With this identification, let $\varphi_E = \varphi|_{L^p(\mu_E)}$. Then $\varphi_E$ is a bounded linear functional on $L^p(\mu_E)$ and $\|\varphi_E\| \le \|\varphi\|$.

If $E \in \mathcal{S}$ and $\mu(E) < \infty$, then the finite measure case that we have already proved as applied to $\varphi_E$ implies that there exists a unique $h_E \in L^{p'}(\mu_E)$ such that

9.46 $$\varphi(f) = \int_E f h_E \, d\mu \quad \text{for all } f \in L^p(\mu_E);$$

the uniqueness of $h_E \in L^{p'}(\mu_E)$ holds because the equation $\|\varphi_h\| = \|h\|_{p'}$ implies that the difference of two different choices for $h_E$ will have norm 0 in $L^{p'}(\mu_E)$. This uniqueness implies that if $D, E \in \mathcal{S}$ and $D \subset E$, then $h_D(x) = h_E(x)$ for almost every $x \in D$.

For each $k \in \mathbf{Z}^+$, there exists $f_k \in L^p(\mu)$ such that

9.47 $$\|f_k\|_p \leq 1 \quad \text{and} \quad |\varphi(f_k)| > \|\varphi\| - \tfrac{1}{k}.$$

The Dominated Convergence Theorem (3.30) implies that

$$\lim_{n \to \infty} \big\|f_k \chi_{\{x \in X : |f_k(x)| > \frac{1}{n}\}} - f_k\big\|_p = 0$$

for each $k \in \mathbf{Z}^+$. Thus we can replace $f_k$ by $f_k \chi_{\{x \in X : |f_k(x)| > \frac{1}{n}\}}$ for sufficiently large $n$ and still have 9.47 hold. Set $D_k = \{x \in X : |f_k(x)| > \frac{1}{n}\}$. Then $\mu(D_k) < \infty$ [because $f_k \in L^p(\mu)$] and with the new function $f_k$ we have

9.48 $$f_k(x) = 0 \text{ for all } x \in X \setminus D_k.$$

Let $E_k = D_1 \cup \cdots \cup D_k$. Because $E_1 \subset E_2 \subset \cdots$, we see that if $j < k$, then $h_{E_j}(x) = h_{E_k}(x)$ for almost every $x \in E_j$. Also, 9.47 and 9.48 imply that

9.49 $$\lim_{k \to \infty} \|h_{E_k}\|_{p'} = \lim_{k \to \infty} \|\varphi_{E_k}\| = \|\varphi\|.$$

Let $E = \bigcup_{k=1}^{\infty} E_k$. Let $h$ be the function that equals $h_k$ almost everywhere on $E_k$ for each $k \in \mathbf{Z}^+$ and equals 0 on $X \setminus E$. The Monotone Convergence Theorem and 9.49 show that $\|h\|_{p'} = \|\varphi\|$.

If $f \in L^p(\mu_E)$, then $\lim_{k \to \infty} \|f - f\chi_{E_k}\|_p = 0$ by the Dominated Convergence Theorem. Thus if $f \in L^p(\mu_E)$, then

9.50 $$\varphi(f) = \lim_{k \to \infty} \varphi(f\chi_{E_k}) = \lim_{k \to \infty} \int f\chi_{E_k} h \, d\mu = \int fh \, d\mu,$$

where the first equality follows from the continuity of $\varphi$, the second equality follows from 9.46 as applied to each $E_k$ [valid because $\mu(E_k) < \infty$], and the third equality follows from an application of Hölder's Inequality.

If $D$ is an $\mathcal{S}$-measurable subset of $X \setminus E$ with $\mu(D) < \infty$, then $\|h_D\|_{p'} = 0$ because otherwise we would have $\|h + h_D\|_{p'} > \|h\|_{p'}$ and the linear functional on $L^p(\mu)$ induced by $h + h_D$ would have norm larger than $\|\varphi\|$ even though it agrees with $\varphi$ on $L^p(\mu_{E \cup D})$. Because $\|h_D\|_{p'} = 0$, we see from 9.50 that $\varphi(f) = \int fh \, d\mu$ for all $f \in L^p(\mu_{E \cup D})$.

Every element of $L^p(\mu)$ can be approximated in norm by elements of $L^p(\mu_E)$ plus functions that live on subsets of $X \setminus E$ with finite measure. Thus $\varphi(f) = \int fh \, d\mu$ for all $f \in L^p(\mu)$, completing the proof. ■

# EXERCISES 9B

**1** Suppose $\nu$ is a real measure on a measurable space $(X, \mathcal{S})$. Prove that the Hahn decomposition of $\nu$ is almost unique, in the sense that if $A, B$ and $A', B'$ are pairs satisfying the Hahn Decomposition Theorem (9.23), then

$$|\nu|(A \setminus A') = |\nu|(A' \setminus A) = |\nu|(B \setminus B') = |\nu|(B' \setminus B) = 0.$$

**2** Suppose $\mu$ is a (positive) measure and $g, h \in \mathcal{L}^1(\mu)$. Prove that $g\, d\mu \perp h\, d\mu$ if and only if $g(x)h(x) = 0$ for almost every $x \in X$.

**3** Suppose $\nu$ and $\mu$ are complex measures on a measurable space $(X, \mathcal{S})$. Show that the following are equivalent:

- $\nu \perp \mu$;
- $|\nu| \perp |\mu|$;
- $\operatorname{Re} \nu \perp \operatorname{Re} \mu$ and $\operatorname{Im} \nu \perp \operatorname{Im} \mu$.

**4** Suppose $\nu$ and $\mu$ are complex measures on a measurable space $(X, \mathcal{S})$. Prove that if $\nu \perp \mu$, then $|\nu + \mu| = |\nu| + |\mu|$ and $\|\nu + \mu\| = \|\nu\| + \|\mu\|$.

**5** Suppose $\nu$ and $\mu$ are finite (positive) measures. Prove that $\nu \perp \mu$ if and only if $\|\nu - \mu\| = \|\nu\| + \|\mu\|$.

**6** Suppose $\mu$ is a complex or positive measure on a measurable space $(X, \mathcal{S})$. Prove that

$$\{\nu \in \mathcal{M}_{\mathbf{F}}(\mathcal{S}) : \nu \perp \mu\}$$

is a closed subspace of $\mathcal{M}_{\mathbf{F}}(\mathcal{S})$.

**7** Use the Cantor set to prove that there exists a (positive) measure $\nu$ on $(\mathbf{R}, \mathcal{B})$ such that $\nu \perp \lambda$ and $\nu(\mathbf{R}) \neq 0$ but $\nu(\{x\}) = 0$ for every $x \in \mathbf{R}$; here $\lambda$ denotes Lebesgue measure on the $\sigma$-algebra $\mathcal{B}$ of Borel subsets of $\mathbf{R}$.
[*The second bullet point in Example 9.29 does not provide an example of the desired behavior because in that example,* $\nu(\{r_k\}) \neq 0$ *for all* $k \in \mathbf{Z}^+$ *with* $w_k \neq 0$.]

**8** Suppose $\nu$ is a real measure on a measurable space $(X, \mathcal{S})$. Prove that

$$\nu^+(E) = \sup\{\nu(D) : D \in \mathcal{S} \text{ and } D \subset E\}$$

and

$$\nu^-(E) = -\inf\{\nu(D) : D \in \mathcal{S} \text{ and } D \subset E\}.$$

**9** Suppose $\mu$ is a (positive) finite measure on a measurable space $(X, \mathcal{S})$ and $h$ is a nonnegative function in $\mathcal{L}^1(\mu)$. Thus $h\, d\mu \ll d\mu$. Find a reasonable condition on $h$ that is equivalent to the condition $d\mu \ll h\, d\mu$.

 

**10** Suppose $\mu$ is a (positive) measure on a measurable space $(X, \mathcal{S})$ and $\nu$ is a complex measure on $(X, \mathcal{S})$. Show that the following are equivalent:

- $\nu \ll \mu$;
- $|\nu| \ll \mu$;
- $\operatorname{Re} \nu \ll \mu$ and $\operatorname{Im} \nu \ll \mu$.

**11** Suppose $\mu$ is a (positive) measure on a measurable space $(X, \mathcal{S})$ and $\nu$ is a real measure on $(X, \mathcal{S})$. Show that $\nu \ll \mu$ if and only if $\nu^+ \ll \mu$ and $\nu^- \ll \mu$.

**12** Suppose $\mu$ is a (positive) measure on a measurable space $(X, \mathcal{S})$. Prove that

$$\{\nu \in \mathcal{M}_{\mathbf{F}}(\mathcal{S}) : \nu \ll \mu\}$$

is a closed subspace of $\mathcal{M}_{\mathbf{F}}(\mathcal{S})$.

**13** Give an example to show that the Radon–Nikodym Theorem (9.36) can fail if the $\sigma$-finite hypothesis is eliminated.

**14** Suppose $\mu$ is a (positive) $\sigma$-finite measure on a measurable space $(X, \mathcal{S})$ and $\nu$ is a complex measure on $(X, \mathcal{S})$. Show that the following are equivalent:

- $\nu \ll \mu$;
- for every $\varepsilon > 0$, there exists $\delta > 0$ such that $|\nu(E)| < \varepsilon$ for every set $E \in \mathcal{S}$ with $\mu(E) < \delta$.
- for every $\varepsilon > 0$, there exists $\delta > 0$ such that $|\nu|(E) < \varepsilon$ for every set $E \in \mathcal{S}$ with $\mu(E) < \delta$.

**15** Show that if $p = 1$, then 9.42 can fail without the extra hypothesis that $\mu$ is a $\sigma$-finite (positive) measure.

**16** Prove 9.42 [with the extra hypothesis that $\mu$ is a $\sigma$-finite (positive) measure] in the case where $p = 1$.

**17** Explain where the proof of 9.42 fails if $p = \infty$.

**18** Prove that if $\mu$ is a (positive) measure and $1 < p < \infty$, then $L^p(\mu)$ is reflexive. [*See the definition before Exercise 17 in Section 7B for the meaning of* reflexive.]

**19** Prove that $L^1(\mathbf{R})$ is not reflexive.