Chapter

7

L^p Spaces

Fix a measure space $(X, \mathcal{S}, \mu)$ and a positive number p. We begin this chapter by looking at the vector space of measurable functions $f : X \to \mathbf{F}$ such that

$$\int |f|^p \, d\mu < \infty.$$

Important results called Hölder's Inequality and Minkowski's Inequality will help us investigate this vector space. A useful class of Banach spaces appears when we identify functions that differ only on a set of measure 0 and require $p \geq 1$.

The main building of the Swiss Federal Institute of Technology (ETH Zürich). Hermann Minkowski (1864–1909) taught at this university from 1896 to 1902. During this time, Albert Einstein (1879–1955) was a student in several of Minkowski's mathematics classes. Minkowski later created mathematics that helped explain Einstein's special theory of relativity.

7A $\mathcal{L}^p(\mu)$

Hölder's Inequality

Our next major goal is to define an important class of vector spaces that generalize the vector spaces $\mathcal{L}^1(\mu)$ and ℓ^1 introduced in the last two bullet points of Example 6.14. We begin this process with the definition below. The terminology *p-norm* introduced below is convenient even though it is not necessarily a norm.

7.1 Definition $\|f\|_p$

Suppose $(X, \mathcal{S}, \mu)$ is a measure space, $p \in (0, \infty)$, and $f\colon X \to \mathbf{F}$ is $\mathcal{S}$-measurable. Then the *p-norm* of f is denoted by $\|f\|_p$ and is defined by

$$\|f\|_p = \Big(\int |f|^p \, d\mu\Big)^{1/p}.$$

Also, $\|f\|_\infty$, which is called the *essential supremum* of f, is defined by

$$\|f\|_\infty = \inf\{t > 0 : \mu(\{x \in X : |f(x)| > t\}) = 0\}.$$

The exponent $1/p$ appears in the definition of the p-norm $\|f\|_p$ because we want the equation $\|\alpha f\|_p = |\alpha| \, \|f\|_p$ to hold.

For $p \in (0, \infty)$, the p-norm $\|f\|_p$ does not change if f changes on a set of μ-measure 0. By using the essential supremum rather than the supremum in the definition of $\|f\|_\infty$, we arrange for the ∞-norm $\|f\|_\infty$ to enjoy this same property. Also, Exercise 16 in this section shows why using the essential supremum rather than the supremum is the right definition.

7.2 Example *p-norm for counting measure*

Suppose μ is counting measure on $\mathbf{Z}^+$. If $a = (a_1, a_2, \dots)$ is a sequence in $\mathbf{F}$ and $p \in (0, \infty)$, then

$$\|a\|_p = \Big(\sum_{k=1}^{\infty} |a_k|^p\Big)^{1/p} \quad \text{and} \quad \|a\|_\infty = \sup\{|a_k| : k \in \mathbf{Z}^+\}.$$

Note that for counting measure, the essential supremum and the supremum are the same because in this case there are no sets of measure 0 other than the empty set.

Now we can define our generalization of $\mathcal{L}^1(\mu)$, which was defined in the last bullet point of Example 6.14.

7.3 Definition $\mathcal{L}^p(\mu)$

Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $p \in (0, \infty]$. The Lebesgue space $\mathcal{L}^p(\mu)$, sometimes denoted $\mathcal{L}^p(X, \mathcal{S}, \mu)$, is defined to be the set of $\mathcal{S}$-measurable functions $f\colon X \to \mathbf{F}$ such that $\|f\|_p < \infty$.

7.4 Example ℓ^p

When μ is counting measure on $\mathbf{Z}^+$, the set $\mathcal{L}^p(\mu)$ is often denoted by ℓ^p (pronounced *little ell-p*). Thus if $p \in (0, \infty)$, then

$$\ell^p = \{(a_1, a_2, \dots) : \text{each } a_k \in \mathbf{F} \text{ and } \sum_{k=1}^{\infty} |a_k|^p < \infty\}$$

and

$$\ell^\infty = \{(a_1, a_2, \dots) : \text{each } a_k \in \mathbf{F} \text{ and } \sup_{k \in \mathbf{Z}^+} |a_k| < \infty\}.$$

Inequality 7.5(a) below will provide an easy proof that $\mathcal{L}^p(\mu)$ is closed under addition. Soon we will prove Minkowski's Inequality (7.14), which provides an important improvement of 7.5(a) when $p \geq 1$ but is more complicated to prove.

7.5 *$\mathcal{L}^p(\mu)$ is a vector space*

Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $p \in (0, \infty)$. Then

(a)
$$\|f + g\|_p^p \leq 2^p (\|f\|_p^p + \|g\|_p^p)$$

and

(b)
$$\|\alpha f\|_p = |\alpha| \, \|f\|_p$$

for all $f, g \in \mathcal{L}^p(\mu)$ and all $\alpha \in \mathbf{F}$. Furthermore, with the usual operations of addition and scalar multiplication of functions, $\mathcal{L}^p(\mu)$ is a vector space.

Proof Suppose $f, g \in \mathcal{L}^p(\mu)$. If $x \in X$, then

$$\begin{aligned} |f(x) + g(x)|^p &\leq (|f(x)| + |g(x)|)^p \\ &\leq (2 \max\{|f(x)|, |g(x)|\})^p \\ &\leq 2^p (|f(x)|^p + |g(x)|^p). \end{aligned}$$

Integrating both sides of the inequality above with respect to μ gives the desired inequality

$$\|f + g\|_p^p \leq 2^p (\|f\|_p^p + \|g\|_p^p).$$

This inequality implies that if $\|f\|_p < \infty$ and $\|g\|_p < \infty$, then $\|f + g\|_p < \infty$. Thus $\mathcal{L}^p(\mu)$ is closed under addition.

The proof that

$$\|\alpha f\|_p = |\alpha| \, \|f\|_p$$

follows easily from the definition of $\|\cdot\|_p$. This equality implies that $\mathcal{L}^p(\mu)$ is closed under scalar multiplication.

Because $\mathcal{L}^p(\mu)$ contains the constant function 0 and is closed under addition and scalar multiplication, $\mathcal{L}^p(\mu)$ is a subspace of $\mathbf{F}^X$ and thus is a vector space. ■

What we call the *dual exponent* in the definition below is often called the *conjugate exponent* or the *conjugate index*. However, the terminology *dual exponent* conveys more meaning because of results (7.25 and 7.26) that we will see in the next section.

7.6 Definition ***dual exponent;*** p'

For $p \in [1, \infty]$ the *dual exponent* of p is denoted by p' and is the element of $[1, \infty]$ such that

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

7.7 Example *dual exponents*

$$1' = \infty, \quad \infty' = 1, \quad 2' = 2, \quad 4' = 4/3, \quad (4/3)' = 4$$

The result below will be a key tool in proving Hölder's Inequality (7.9).

7.8 ***Young's Inequality***

Suppose $p \in (1, \infty)$. Then

$$ab \leq \frac{a^p}{p} + \frac{b^{p'}}{p'}$$

for all $a \geq 0$ and $b \geq 0$.

Proof Fix $b > 0$ and define a function $f \colon (0, \infty) \to \mathbf{R}$ by

$$f(a) = \frac{a^p}{p} + \frac{b^{p'}}{p'} - ab.$$

British mathematician William Henry Young (1863–1942) published what is now called Young's Inequality in 1912.

Thus $f'(a) = a^{p-1} - b$. Hence f is decreasing on the interval $\bigl(0, b^{1/(p-1)}\bigr)$ and f is increasing on the interval $\bigl(b^{1/(p-1)}, \infty\bigr)$. Thus f has a global minimum at $b^{1/(p-1)}$. A tiny bit of arithmetic [use $p/(p-1) = p'$] shows that $f\bigl(b^{1/(p-1)}\bigr) = 0$. Thus $f(a) \geq 0$ for all $a \in (0, \infty)$, which implies the desired inequality. ■

The important result below furnishes a key tool that is used in the proof of Minkowski's Inequality (7.14).

7.9 ***Hölder's Inequality***

Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $f, h \colon X \to \mathbf{F}$ are $\mathcal{S}$-measurable. Then

$$\|fh\|_1 \leq \|f\|_p \, \|h\|_{p'}$$

for all $p \in [1, \infty]$.

Proof We fix $p \in (1, \infty)$, leaving the cases $p = 1$ and $p = \infty$ as exercises for the reader.

First consider the special case where $\|f\|_p = \|h\|_{p'} = 1$. Young's Inequality (7.8) tells us that

$$|f(x)h(x)| \leq \frac{|f(x)|^p}{p} + \frac{|h(x)|^{p'}}{p'}$$

for all $x \in X$. Integrating both sides of the inequality above with respect to μ shows that $\|fh\|_1 \leq 1 = \|f\|_p \, \|h\|_{p'}$, completing the proof in this special case.

Hölder's Inequality was proved in 1889 by German mathematician Otto Hölder (1859–1937).

If $\|f\|_p = 0$ or $\|h\|_{p'} = 0$, then $\|fh\|_1 = 0$ and the desired inequality holds. Similarly, if $\|f\|_p = \infty$ or $\|h\|_{p'} = \infty$, then the desired inequality clearly holds.Thus we will assume that $0 < \|f\|_p < \infty$ and $0 < \|h\|_{p'} < \infty$.

Now define $\mathcal{S}$-measurable functions $f_1, h_1 \colon X \to \mathbf{F}$ by

$$f_1 = \frac{f}{\|f\|_p} \quad \text{and} \quad h_1 = \frac{h}{\|h\|_{p'}}.$$

Then $\|f_1\|_p = 1$ and $\|h_1\|_{p'} = 1$. By the result for our special case, we have $\|f_1 h_1\|_1 \leq 1$, which implies that $\|fh\|_1 \leq \|f\|_p \, \|h\|_{p'}$. ■

The next result gives a key containment among Lebesgue spaces with respect to a finite measure. Note the crucial role that Hölder's Inequality plays in the proof.

7.10 $\mathcal{L}^q(\mu) \subset \mathcal{L}^p(\mu)$ ***if*** $p < q$ ***and*** $\mu(X) < \infty$

Suppose $(X, \mathcal{S}, \mu)$ is a finite measure space and $0 < p < q < \infty$. Then

$$\|f\|_p \leq \mu(X)^{(q-p)/(pq)} \|f\|_q$$

for all $f \in \mathcal{L}^q(\mu)$. Furthermore, $\mathcal{L}^q(\mu) \subset \mathcal{L}^p(\mu)$.

Proof Fix $f \in \mathcal{L}^q(\mu)$. Let $r = \frac{q}{p}$. Thus $r > 1$. A short calculation shows that $r' = \frac{q}{q-p}$. Now Hölder's Inequality (7.9) with p replaced by r and f replaced by $|f|^p$ and h replaced by the constant function 1 gives

$$\begin{aligned}\int |f|^p \, d\mu &\leq \Bigl(\int (|f|^p)^r \, d\mu\Bigr)^{1/r} \Bigl(\int 1^{r'} \, d\mu\Bigr)^{1/r'} \\ &= \mu(X)^{(q-p)/q} \Bigl(\int |f|^q \, d\mu\Bigr)^{p/q}.\end{aligned}$$

Now raise both sides of the inequality above to the power $\frac{1}{p}$, getting

$$\Bigl(\int |f|^p \, d\mu\Bigr)^{1/p} \leq \mu(X)^{(q-p)/(pq)} \Bigl(\int |f|^q \, d\mu\Bigr)^{1/q},$$

which is the desired inequality.

The inequality above shows that $f \in \mathcal{L}^p(\mu)$. Thus $\mathcal{L}^q(\mu) \subset \mathcal{L}^p(\mu)$. ■

7.11 Example $\mathcal{L}^p(E)$

We adopt the common convention that if E is a Borel (or Lebesgue measurable) subset of $\mathbf{R}$ and $p \in (0, \infty]$, then $\mathcal{L}^p(E)$ means $\mathcal{L}^p(\lambda_E)$, where λ_E denotes Lebesgue measure λ restricted to the Borel (or Lebesgue measurable) subsets of $\mathbf{R}$ that are contained in E.

With this convention, 7.10 implies that

$$\text{if } 0 < p < q < \infty, \text{ then } \mathcal{L}^q([0,1]) \subset \mathcal{L}^p([0,1] \text{ and } \|f\|_p \le \|f\|_q$$

for $f \in \mathcal{L}^q([0,1]$. See Exercises 12 and 13 in this section for further results concerning $\mathcal{L}^p([0,1])$.

Minkowski's Inequality

The next result will be used as a tool to prove Minkowski's Inequality (7.14). Once again, note the crucial role that Hölder's Inequality plays in the proof.

7.12 ***Formula for*** $\|f\|_p$

Suppose $(X, \mathcal{S}, \mu)$ is a measure space, $p \in [1, \infty)$, and $f \in \mathcal{L}^p(\mu)$. Then

$$\|f\|_p = \sup\Big\{\Big|\int fh\,d\mu\Big| : h \in \mathcal{L}^{p'}(\mu) \text{ and } \|h\|_{p'} \le 1\Big\}.$$

Proof If $\|f\|_p = 0$, then both sides of the equation in the conclusion of this result equal 0. Thus we will also assume that $\|f\|_p \neq 0$.

Hölder's Inequality (7.9) implies that if $h \in \mathcal{L}^{p'}(\mu)$ and $\|h\|_{p'} \le 1$, then

$$\Big|\int fh\,d\mu\Big| \le \int |fh|\,d\mu \le \|f\|_p\,\|h\|_{p'} \le \|f\|_p.$$

Thus $\sup\big\{\big|\int fh\,d\mu\big| : h \in \mathcal{L}^{p'}(\mu) \text{ and } \|h\|_{p'} \le 1\big\} \le \|f\|_p$.

To prove the inequality in the other direction, define $h \colon X \to \mathbf{F}$ by

$$h(x) = \frac{\overline{f(x)}\,|f(x)|^{p-2}}{\|f\|_p^{p/p'}} \qquad (\text{set } h(x) = 0 \text{ when } f(x) = 0.)$$

Then $\int fh\,d\mu = \|f\|_p$ and $\|h\|_{p'} = 1$, as you should verify [use $p - \frac{p}{p'} = 1$]. Thus $\|f\|_p \le \sup\big\{\big|\int fh\,d\mu\big| : h \in \mathcal{L}^{p'}(\mu) \text{ and } \|h\|_{p'} \le 1\big\}$, as desired. ■

7.13 Example *a point with infinite measure*

Suppose X is a set with exactly one element b and μ is the measure such that $\mu(\varnothing) = 0$ and $\mu(\{b\}) = \infty$. Then $\mathcal{L}^1(\mu)$ consists only of the 0 function. Thus if $p = \infty$ and f is the function whose value at b equals 1, then $\|f\|_\infty = 1$ but the right side of the equation in 7.12 equals 0. Thus 7.12 can fail when $p = \infty$.

Example 7.13 shows that we cannot take $p = \infty$ in 7.12. However, if μ is a σ-finite measure, then 7.12 holds even when $p = \infty$; see Exercise 8.

The next result, which is called Minkowski's Inequality, is an improvement for $p \geq 1$ of the inequality 7.5(a). For the situation when $p \in (0, 1)$, see Exercise 18.

7.14 *Minkowski's Inequality*

Suppose $(X, \mathcal{S}, \mu)$ is a measure space, $p \in [1, \infty]$, and $f, g \in \mathcal{L}^p(\mu)$. Then

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p.$$

Proof Assume that $1 \leq p < \infty$ (the case $p = \infty$ is left as an exercise for the reader). Inequality 7.5(a) implies that $f + g \in \mathcal{L}^p(\mu)$.

Suppose $h \in \mathcal{L}^{p'}(\mu)$ and $\|h\|_{p'} \leq 1$. Then

$$\begin{aligned} \Big|\int (f+g)h \, d\mu\Big| &\leq \int |fh| \, d\mu + \int |gh| \, d\mu \\ &\leq \|f\|_p \, \|h\|_{p'} + \|g\|_p \, \|h\|_{p'} \\ &\leq \|f\|_p + \|g\|_p, \end{aligned}$$

where the second line comes comes from Hölder's Inequality (7.9). Now take the supremum in the inequality above over the set of $h \in \mathcal{L}^{p'}(\mu)$ such that $\|h\|_{p'} \leq 1$. By 7.12, we get $\|f + g\|_p \leq \|f\|_p + \|g\|_p$, as desired. ∎

EXERCISES 7A

1 Suppose μ is a measure. Prove that

$$\|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty \quad \text{and} \quad \|\alpha f\|_\infty = |\alpha| \, \|f\|_\infty$$

for all $f, g \in \mathcal{L}^\infty(\mu)$ and all $\alpha \in \mathbf{F}$. Conclude that with the usual operations of addition and scalar multiplication of functions, $\mathcal{L}^\infty(\mu)$ is a vector space.

2 Suppose $a \geq 0$, $b \geq 0$, and $p \in (1, \infty)$. Prove that Young's Inequality (7.8)

$$ab \leq \frac{a^p}{p} + \frac{b^{p'}}{p'}$$

is an equality if and only if $a^p = b^{p'}$.

3 Suppose $a_1, \dots, a_n$ are nonnegative numbers. Prove that

$$(a_1 + \cdots + a_n)^5 \leq n^4({a_1}^5 + \cdots + {a_n}^5).$$

4 Prove Hölder's Inequality (7.9) in the cases $p = 1$ and $p = \infty$.

5 Suppose $(X, \mathcal{S}, \mu)$ is a measure space, $p \in (1, \infty)$, and $f \in \mathcal{L}^p(\mu)$, $h \in \mathcal{L}^{p'}(\mu)$. Prove that Hölder's Inequality (7.9) is an equality if and only if there exist nonnegative numbers a and b, not both 0, such that

$$a|f(x)|^p = b|h(x)|^{p'}$$

for almost every $x \in X$.

6 Suppose $(X, \mathcal{S}, \mu)$ is a measure space, $f \in \mathcal{L}^1(\mu)$, and $h \in \mathcal{L}^\infty(\mu)$. Prove that $\|fh\|_1 = \|f\|_1 \, \|h\|_\infty$ (which is Hölder's Inequality with $p = 1$) is an equality if and only if

$$|h(x)| = \|h\|_\infty$$

for almost every $x \in X$ such that $f(x) \neq 0$.

7 Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $f, h \colon X \to \mathbf{F}$ are $\mathcal{S}$-measurable. Prove that

$$\|fh\|_r \leq \|f\|_p \, \|h\|_q$$

for all positive numbers p, q, r such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$.

8 Show that the formula in 7.12 holds for $p = \infty$ if μ is a σ-finite measure.

9 Suppose $(X, \mathcal{S}, \mu)$ is a σ-finite measure space and $p \in [1, \infty]$. Prove that if $f \colon X \to \mathbf{F}$ is an $\mathcal{S}$-measurable function such that $fh \in \mathcal{L}^1(\mu)$ for every $h \in \mathcal{L}^{p'}(\mu)$, then $f \in \mathcal{L}^p(\mu)$.

10 Suppose $0 < p < q \leq \infty$.

(a) Prove that $\ell^p \subset \ell^q$.

(b) Prove that $\|(a_1, a_2, \ldots)\|_p \geq \|(a_1, a_2, \ldots)\|_q$ for every sequence $a_1, a_2, \ldots$ of elements of $\mathbf{F}$.

11 Show that $\displaystyle\bigcap_{p>1} \ell^p \neq \ell^1$.

12 Show that $\displaystyle\bigcap_{p<\infty} \mathcal{L}^p([0,1]) \neq \mathcal{L}^\infty([0,1])$.

13 Show that $\displaystyle\bigcup_{p>1} \mathcal{L}^p([0,1]) \neq \mathcal{L}^1([0,1])$.

14 Suppose $p, q \in (0, \infty]$, with $p \neq q$. Prove that neither of the sets $\mathcal{L}^p(\mathbf{R})$ and $\mathcal{L}^q(\mathbf{R})$ is a subset of the other.

15 Show that there exists $f \in \mathcal{L}^2(\mathbf{R})$ such that $f \notin \mathcal{L}^p(\mathbf{R})$ for all $p \in (0, \infty] \setminus \{2\}$.

16 Suppose $(X, \mathcal{S}, \mu)$ is a finite measure space. Prove that

$$\lim_{p \to \infty} \|f\|_p = \|f\|_\infty$$

for every $\mathcal{S}$-measurable function $f \colon X \to \mathbf{F}$.

17 Suppose $(X, \mathcal{S}, \mu)$ is a measure space, $p \in (1, \infty)$, and $f, g \in \mathcal{L}^p(\mu)$. Prove that Minkowski's Inequality (7.14) is an equality if and only if there exist nonnegative numbers a and b, not both 0, such that

$$af(x) = bg(x)$$

for almost every $x \in X$.

18 (a) Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $p \in (0, 1)$. Show that

$$\|f + g\|_p \geq \|f\|_p + \|g\|_p$$

for all $\mathcal{S}$-measurable functions $f, g\colon X \to [0, \infty)$.

(b) Give an example to show that the reverse Minkowski inequality in part (a) does not necessarily hold without the hypothesis that f and g are nonnegative functions.

19 Suppose μ is a measure, $p \in (0, \infty]$, and $f \in \mathcal{L}^p(\mu)$. Prove that for every $\varepsilon > 0$, there exists a simple function $g \in \mathcal{L}^p(\mu)$ such that $\|f - g\|_p < \varepsilon$.
[*This exercise extends 3.48.*]

20 Suppose $p \in (0, \infty)$ and $f \in \mathcal{L}^p(\mathbf{R})$. Prove that for every $\varepsilon > 0$, there exists a step function $g \in \mathcal{L}^p(\mu)$ such that $\|f - g\|_p < \varepsilon$.
[*This exercise extends 3.51.*]

21 Suppose $p \in (0, \infty)$ and $f \in \mathcal{L}^p(\mathbf{R})$. Prove that for every $\varepsilon > 0$, there exists a continuous function $g\colon \mathbf{R} \to \mathbf{R}$ such that $\|f - g\|_p < \varepsilon$ and the set $\{x \in \mathbf{R} : g(x) \neq 0\}$ is bounded.
[*This exercise extends 3.52.*]

22 Suppose $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \nu)$ are σ-finite measure spaces and $0 < p < \infty$. Prove that if $f \in \mathcal{L}^p(\mu \times \nu)$, then

$$[f]_x \in \mathcal{L}^p(\nu) \text{ for almost every } x \in X$$

and

$$[f]^y \in \mathcal{L}^p(\mu) \text{ for almost every } y \in Y,$$

where $[f]_x$ and $[f]^y$ are the cross sections of f as defined in 5.7.

7B $L^p(\mu)$

Definition of $L^p(\mu)$

Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $p \in [1, \infty]$. If there exists a nonempty set $E \in \mathcal{S}$ such that $\mu(E) = 0$, then $\|\chi_E\|_p = 0$ even though $\chi_E \neq 0$; thus $\|\cdot\|_p$ is not a norm on $\mathcal{L}^p(\mu)$. The standard way to deal with this problem is to identify functions that differ only on a set of μ-measure 0. To help make this process more rigorous, we introduce the following definitions.

7.15 Definition $\mathcal{Z}(\mu)$; $\tilde{f}$

Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $p \in (0, \infty]$.

- $\mathcal{Z}(\mu)$ denotes the set of $\mathcal{S}$-measurable functions from X to $\mathbf{F}$ that equal 0 almost everywhere.
- For $f \in \mathcal{L}^p(\mu)$, let $\tilde{f}$ be the subset of $\mathcal{L}^p(\mu)$ defined by

$$\tilde{f} = \{f + z : z \in \mathcal{Z}(\mu)\}.$$

The set $\mathcal{Z}(\mu)$ is clearly closed under scalar multiplication. Also, $\mathcal{Z}(\mu)$ is closed under addition because the union of two sets with μ-measure 0 is a set with μ-measure 0. Thus $\mathcal{Z}(\mu)$ is a subspace of $\mathcal{L}^p(\mu)$, as we had noted in the third bullet point of Example 6.14.

Note that if $f, F \in \mathcal{L}^p(\mu)$, then $\tilde{f} = \tilde{F}$ if and only if $f(x) = F(x)$ for almost every $x \in X$.

7.16 Definition $L^p(\mu)$

Suppose μ is a measure and $p \in (0, \infty]$.

- Let $L^p(\mu)$ denote the collection of subsets of $\mathcal{L}^p(\mu)$ defined by

$$L^p(\mu) = \{\tilde{f} : f \in \mathcal{L}^p(\mu)\}.$$

- For $\tilde{f}, \tilde{g} \in L^p(\mu)$ and $\alpha \in \mathbf{F}$, define $\tilde{f} + \tilde{g}$ and $\alpha\tilde{f}$ by

$$\tilde{f} + \tilde{g} = (f + g)\tilde{} \quad \text{and} \quad \alpha\tilde{f} = (\alpha f)\tilde{}.$$

The last bullet point in the definition above requires a bit of care to verify that it makes sense. The potential problem is that if $\mathcal{Z}(\mu) \neq \{0\}$, then $\tilde{f}$ is not uniquely represented by f. Thus suppose $f, F, g, G \in \mathcal{L}^p(\mu)$ and $\tilde{f} = \tilde{F}$ and $\tilde{g} = \tilde{G}$. For the definition of addition in $L^p(\mu)$ to make sense, we must verify that $(f + g)\tilde{} = (F + G)\tilde{}$. This verification is left to the reader, as is the similar verification that the scalar multiplication defined in the last bullet point above makes sense.

You might want to think of elements of $L^p(\mu)$ as equivalence classes of functions in $\mathcal{L}^p(\mu)$, where two functions are equivalent if they agree almost everywhere.

Note the subtle typographic difference between $\mathcal{L}^p(\mu)$ and $L^p(\mu)$. An element of the calligraphic $\mathcal{L}^p(\mu)$ is a function; an element of the italic $L^p(\mu)$ is a set of functions, any two of which agree almost everywhere.

Mathematicians often pretend that elements of $L^p(\mu)$ are functions, where two functions are considered to be equal if they differ only on a set of μ-measure 0. This fiction is harmless provided that the operations you perform with such "functions" produce the same results if the functions are changed on a set of measure 0.

7.17 Definition $\|\cdot\|_p$ ***on*** $L^p(\mu)$

Suppose μ is a measure and $p \in (0, \infty]$. Define $\|\cdot\|_p$ on $L^p(\mu)$ by

$$\|\tilde{f}\|_p = \|f\|_p$$

for $f \in \mathcal{L}^p(\mu)$.

Note that if $f, F \in \mathcal{L}^p(\mu)$ and $\tilde{f} = \tilde{F}$, then $\|f\|_p = \|F\|_p$. Thus the definition above makes sense.

In the result below, the addition and scalar multiplication on $L^p(\mu)$ come from 7.16 and the norm comes from 7.17.

7.18 $L^p(\mu)$ ***is a normed vector space***

Suppose μ is a measure and $p \in [1, \infty]$. Then $L^p(\mu)$ is a vector space and $\|\cdot\|_p$ is a norm on $L^p(\mu)$.

The proof of the result above is left to the reader, who will surely use Minkowski's Inequality (7.14) to verify the triangle inequality. Note that the additive identity of $L^p(\mu)$ is $\tilde{0}$, which equals $\mathcal{Z}(\mu)$.

If μ is counting measure on $\mathbf{Z}^+$, then

$$\mathcal{L}^p(\mu) = L^p(\mu) = \ell^p$$

because counting measure has no sets of measure 0 other than the empty set.

For readers familiar with quotients of vector spaces: you may recognize that $L^p(\mu)$ is equal to the quotient space

$$\mathcal{L}^p(\mu)/\mathcal{Z}(\mu).$$

For readers who want to learn about quotients of vector spaces: see a textbook for a second course in linear algebra.

The notation introduced below is commonly used in mathematics literature.

7.19 Definition $L^p(E)$ ***for*** $E \subset \mathbf{R}$

If E is a Borel (or Lebesgue measurable) subset of $\mathbf{R}$ and $p \in (0, \infty]$, then $L^p(E)$ means $L^p(\lambda_E)$, where λ_E denotes Lebesgue measure λ restricted to the Borel (or Lebesgue measurable) subsets of $\mathbf{R}$ that are contained in E.

$L^p(\mu)$ is a Banach Space

The proof of the next result contains all the hard work we will need to prove that $L^p(\mu)$ is a Banach space. However, we will state the next result in terms of $\mathcal{L}^p(\mu)$ instead of $L^p(\mu)$ so that we can work with genuine functions. Moving to $L^p(\mu)$ will then be easy (see 7.23).

7.20 *Cauchy sequences in $\mathcal{L}^p(\mu)$ converge*

Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $p \in [1, \infty]$. Suppose $f_1, f_2, \ldots$ is a sequence of functions in $\mathcal{L}^p(\mu)$ such that for every $\varepsilon > 0$, there exists $N \in \mathbf{Z}^+$ such that

$$\|f_j - f_k\|_p < \varepsilon$$

for all $j \geq N$ and $k \geq N$. Then there exists $f \in \mathcal{L}^p(\mu)$ such that

$$\lim_{n\to\infty} \|f_n - f\|_p = 0.$$

Proof The case $p = \infty$ is left as an exercise for the reader. Thus assume $p \in [1, \infty)$.

It suffices to show that $\lim_{m\to\infty} \|f_{n_m} - f\|_p = 0$ for some $f \in \mathcal{L}^p(\mu)$ and some subsequence $f_{n_1}, f_{n_2}, \ldots$ (see Exercise 11 of Section 6B, whose proof does not require the positive definite property of a norm).

Thus dropping to a subsequence (but not relabelling) and setting $f_0 = 0$, we can assume that

$$\sum_{k=1}^{\infty} \|f_k - f_{k-1}\|_p < \infty.$$

Define functions $g_1, g_2, \ldots$ and g from X to $[0, \infty]$ by

$$g_n(x) = \sum_{k=1}^{n} |f_k(x) - f_{k-1}(x)| \quad \text{and} \quad g(x) = \sum_{k=1}^{\infty} |f_k(x) - f_{k-1}(x)|.$$

Minkowski's Inequality (7.14) implies that $\|g_n\|_p \leq \sum_{k=1}^{n} \|f_k - f_{k-1}\|_p$. Clearly $\lim_{n\to\infty} g_n(x) = g(x)$ for every $x \in X$. Thus the Monotone Convergence Theorem (3.17) implies

7.21 $$\int g^p \, d\mu = \lim_{n\to\infty} \int {g_n}^p \, d\mu \leq \Big(\sum_{k=1}^{\infty} \|f_k - f_{k-1}\|_p\Big)^p < \infty.$$

Thus $g(x) < \infty$ for almost every $x \in X$.

Because every infinite series of real numbers that converges absolutely also converges, for almost every $x \in X$ we can define $f(x)$ by

$$f(x) = \sum_{k=1}^{\infty} \big(f_k(x) - f_{k-1}(x)\big) = \lim_{n\to\infty} \sum_{k=1}^{n} \big(f_k(x) - f_{k-1}(x)\big) = \lim_{n\to\infty} f_n(x).$$

In particular, $\lim_{n\to\infty} f_n(x)$ exists for almost every $x \in X$. Define $f(x)$ to be 0 for those $x \in X$ for which the limit does not exist.

We now have a function f that is the pointwise limit (almost everywhere) of $f_1, f_2, \ldots$. The definition of f shows that $|f(x)| \leq g(x)$ for almost every $x \in X$. Thus 7.21 shows that $f \in \mathcal{L}^p(\mu)$.

To show that $\lim_{n\to\infty} \|f_n - f\|_p = 0$, suppose $\varepsilon > 0$ and let $N \in \mathbf{Z}^+$ be such that $\|f_j - f_k\|_p < \varepsilon$ for all $j \geq N$ and $k \geq N$. Suppose $n \geq N$. Then

$$\begin{aligned}\|f_n - f\|_p &= \Big(\int |f_n - f|^p \, d\mu\Big)^{1/p}\\ &\leq \liminf_{m\to\infty} \Big(\int |f_n - f_m|^p \, d\mu\Big)^{1/p}\\ &= \liminf_{m\to\infty} \|f_n - f_m\|_p\\ &\leq \varepsilon,\end{aligned}$$

where the second line above comes from Fatou's Lemma (Exercise 11 in Section 3A). Thus $\lim_{n\to\infty} \|f_n - f\|_p = 0$, as desired. ∎

The proof that we have just completed contains within it the proof of a useful result that is worth stating separately. A sequence can converge in p-norm without converging pointwise anywhere (see, for example, Exercise 10). However, the next result guarantees that some subsequence converges pointwise almost everywhere.

7.22 *Convergent sequences in $\mathcal{L}^p$ have pointwise convergent subsequences*

Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $p \in [1, \infty]$. Suppose $f \in \mathcal{L}^p(\mu)$ and $f_1, f_2, \ldots$ is a sequence of functions in $\mathcal{L}^p(\mu)$ such that $\lim\limits_{n\to\infty} \|f_n - f\|_p = 0$. Then there exists a subsequence $f_{n_1}, f_{n_2}, \ldots$ such that

$$\lim_{m\to\infty} f_{n_m}(x) = f(x)$$

for almost every $x \in X$.

Proof Suppose $f_{n_1}, f_{n_2}, \ldots$ is a subsequence such that

$$\sum_{k=2}^{\infty} \|f_{n_k} - f_{n_{k-1}}\|_p < \infty.$$

An examination of the proof of 7.20 shows that $\lim\limits_{m\to\infty} f_{n_m}(x) = f(x)$ for almost every $x \in X$. ∎

7.23 *$L^p(\mu)$ is a Banach Space*

Suppose μ is a measure and $p \in [1, \infty]$. Then $L^p(\mu)$ is a Banach space.

Proof This result follows immediately from 7.20 and the appropriate definitions. ∎

Duality

In the previous chapter, we gave the special name *linear functionals* to linear maps into the scalar field $\mathbf{F}$. The vector space of bounded linear functionals also gets a special name and a special notation.

7.24 Definition ***dual space***

Suppose V is a normed vector space. Then the *dual* of V, denoted V', is the normed vector space of the bounded linear functionals on V. In other words, $V' = \mathcal{B}(V, \mathbf{F})$.

Because $\mathbf{F}$ is a Banach space, 6.40 implies that the dual of every normed vector space is a Banach space.

In the statement and proof of the next result, an element of an L^p space is denoted by a symbol that makes it look like a function rather than like a collection of functions that agree except on a set of measure 0. However, because integrals and L^p-norms are unchanged when functions change only on a set of measure 0, this notational convenience causes no problems.

7.25 ***Natural map of $L^{p'}(\mu)$ into $\big(L^p(\mu)\big)'$ preserves norms***

Suppose μ is a measure and $p \in (1, \infty]$. For $f \in L^{p'}(\mu)$, define $\varphi_f \colon L^p(\mu) \to \mathbf{F}$ by

$$\varphi_f(h) = \int fh \, d\mu.$$

Then $f \mapsto \varphi_f$ is a one-to-one linear map from $L^{p'}(\mu)$ to $\big(L^p(\mu)\big)'$. Furthermore, $\|\varphi_f\| = \|f\|_{p'}$ for all $f \in L^{p'}(\mu)$.

Proof Suppose $f \in L^{p'}(\mu)$ and $h \in L^p(\mu)$. Then Hölder's Inequality (7.9) tells us that $fh \in L^1(\mu)$ and that $\|fh\|_1 \le \|f\|_{p'} \|h\|_p$. Thus φ_f, as defined above, is a bounded linear map from $L^p(\mu)$ to $\mathbf{F}$. Also, the map $f \mapsto \varphi_f$ is clearly a linear map of $L^{p'}(\mu)$ into $\big(L^p(\mu)\big)'$. Now 7.12 (with the roles of p and p' reversed) shows that

$$\|\varphi_f\| = \sup\{|\varphi_f(h)| : h \in L^p(\mu) \text{ and } \|h\|_p \le 1\} = \|f\|_{p'}.$$

If $f_1, f_2 \in L^{p'}(\mu)$ and $\varphi_{f_1} = \varphi_{f_2}$, then

$$\|f_1 - f_2\|_{p'} = \|\varphi_{f_1 - f_2}\| = \|\varphi_{f_1} - \varphi_{f_2}\| = \|0\| = 0,$$

which implies $f_1 = f_2$. Thus $f \mapsto \varphi_f$ is a one-to-one map from $L^{p'}(\mu)$ to $\big(L^p(\mu)\big)'$. ■

The result in 7.25 fails for some measures μ if $p = 1$. However, if μ is a σ-finite measure, then 7.25 holds even if $p = 1$; see Exercise 13.

Is the range of the map $h \mapsto \varphi_h$ in 7.25 equal to all of $\big(L^p(\mu)\big)'$? The next result provides an affirmative answer to this question in the special case of ℓ^p for $p \in [1, \infty)$. We will deal with this question for more general measures later (see 11.9).

When thinking of ℓ^p as a normed vector space, as in the next result, unless stated otherwise you should always assume that the norm on ℓ^p is the usual norm $\|\cdot\|_p$ that is associated with $\mathcal{L}^p(\mu)$, where μ is counting measure on $\mathbf{Z}^+$.

7.26 *Dual of ℓ^p can be identified with $\ell^{p'}$*

Suppose $p \in [1, \infty)$. For $h \in \ell^{p'}$, define $\varphi_h \colon \ell^p \to \mathbf{F}$ by

$$\varphi_h(f) = \sum_{k=1}^{\infty} f_k h_k.$$

Then $h \mapsto \varphi_h$ is a one-to-one linear map from $\ell^{p'}$ onto $(\ell^p)'$. Furthermore, $\|\varphi_h\| = \|h\|_{p'}$ for all $h \in \ell^{p'}$.

Proof For $k \in \mathbf{Z}^+$, let $e_k \in \ell^p$ be the sequence in which each term equals 0 except that the k^{th} term equals 1; thus $e_k = (0, \dots, 0, 1, 0, \dots)$.

Suppose $\varphi \in (\ell^p)'$. Define a sequence $h = (h_1, h_2, \dots)$ of numbers in $\mathbf{F}$ by

$$h_k = \varphi(e_k).$$

Suppose $f = (f_1, f_2, \dots) \in \ell^p$. Then

$$f = \sum_{k=1}^{\infty} f_k e_k,$$

where the infinite sum converges in the norm of ℓ^p (the proof would fail here if we allowed p to equal ∞). Because φ is a bounded linear functional on ℓ^p, applying φ to both sides of the equation above shows that

$$\varphi(f) = \sum_{k=1}^{\infty} f_k h_k.$$

We still need to prove that $h \in \ell^{p'}$. To do this, for $N \in \mathbf{Z}^+$ let μ_N be counting measure on $\{1, 2, \dots, N\}$. We can think of $L^p(\mu_N)$ as a subspace of ℓ^p by identifying each $(f_1, \dots, f_N) \in L^p(\mu_N)$ with $(f_1, \dots, f_N, 0, 0, \dots) \in \ell^p$. Restricting the linear functional φ to $L^p(\mu_N)$ gives the linear functional on $L^p(\mu_N)$ that satisfies the following equation:

$$\varphi|_{L^p(\mu_N)}(f_1, \dots, f_N) = \sum_{k=1}^{N} f_k h_k.$$

Now 7.25 (also see Exercise 13 for the case where $p = 1$) gives

$$\begin{aligned} \|(h_1, \dots, h_N)\|_{p'} &= \|\varphi|_{L^p(\mu_N)}\| \\ &\leq \|\varphi\|. \end{aligned}$$

Because $\lim_{N \to \infty} \|(h_1, \dots, h_N)\|_{p'} = \|h\|_{p'}$, the inequality above implies that $\|h\|_{p'} \leq \|\varphi\|$. Thus $h \in \ell^{p'}$, which implies that $\varphi = \varphi_h$, completing the proof. ∎

The previous result does not hold when $p = \infty$. In other words, the dual of ℓ^∞ cannot be identified with ℓ^1. However, see Exercise 14, which shows that the dual of a natural subspace of ℓ^∞ can be identified with ℓ^1.

EXERCISES 7B

1 Suppose $N > 1$ and $p \in (0,1)$. Prove that if $\|\cdot\|$ is defined on $\mathbf{F}^N$ by

$$\|(a_1,\ldots,a_N)\| = \big(|a_1|^p + \cdots + |a_N|^p\big)^{1/p},$$

then $\|\cdot\|$ is not a norm on $\mathbf{F}^N$.

2 (a) Suppose $1 \le p < \infty$. Prove that there is a countable subset of ℓ^p whose closure equals ℓ^p.

(b) Prove that there does not exist a countable subset of ℓ^∞ whose closure equals ℓ^∞.

3 (a) Suppose $1 \le p < \infty$. Prove that there is a countable subset of $L^p(\mathbf{R})$ whose closure equals $L^p(\mathbf{R})$.

(b) Prove that there does not exist a countable subset of $L^\infty(\mathbf{R})$ whose closure equals $L^\infty(\mathbf{R})$.

4 (a) Prove that if μ is a measure, $p \in (1,\infty)$, and $f, g \in L^p(\mu)$ are such that

$$\|f\|_p = \|g\|_p = \left\|\frac{f+g}{2}\right\|_p,$$

then $f = g$.

(b) Give an example to show that the result in part (a) can fail if $p = 1$.

(c) Give an example to show that the result in part (a) can fail if $p = \infty$.

5 Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $p \in (0,1)$. Show that

$$\|f+g\|_p^p \le \|f\|_p^p + \|g\|_p^p$$

for all $\mathcal{S}$-measurable functions $f, g\colon X \to \mathbf{F}$.

6 Prove that $L^p(\mu)$, with addition and scalar multiplication as defined in 7.16 and norm defined as in 7.17, is a normed vector space. In other words, prove 7.18.

7 Prove 7.20 for the case $p = \infty$.

8 Prove that 7.20 also holds for $p \in (0,1)$.

9 Prove that 7.22 also holds for $p \in (0,1)$.

10 Prove that there exists a sequence $f_1, f_2, \ldots$ of functions in $\mathcal{L}^1([0,1])$ such that $\lim\limits_{n\to\infty} \|f_n\|_1 = 0$ but

$$\sup\{f_n(x) : n \in \mathbf{Z}^+\} = \infty$$

for every $x \in [0,1]$.

11 Suppose $(X, \mathcal{S}, \mu)$ is a measure space, $p \in [1, \infty]$, $f \in \mathcal{L}^p(\mu)$, and $f_1, f_2, \ldots$ is a sequence in $\mathcal{L}^p(\mu)$ such that $\lim_{n\to\infty} \|f_n - f\|_p = 0$. Show that if $g\colon X \to \mathbf{F}$ is a function such that $\lim_{n\to\infty} f_n(x) = g(x)$ for almost all $x \in X$, then $f(x) = g(x)$ for almost all $x \in X$.

12 Suppose $p \in [1, \infty]$. Prove that

$$\{(a_1, a_2, \ldots) \in \ell^p : a_k \neq 0 \text{ for every } k \in \mathbf{Z}^+\}$$

is not an open subset of ℓ^p.

13 (a) Give an example of a measure μ such that 7.25 fails for $p = 1$.

(b) Show that if μ is a σ-finite measure, then 7.25 holds for $p = 1$.

14 Let

$$c_0 = \{(f_1, f_2, \ldots) \in \ell^\infty : \lim_{n\to\infty} f_n = 0\}.$$

Give c_0 the norm that it inherits as a subspace of ℓ^∞.

(a) Prove that c_0 is a Banach space.

(b) Prove that the dual of c_0 can be identified with ℓ^1.

15 Suppose $1 \leq p \leq 2$.

(a) Prove that if $w, z \in \mathbf{C}$, then

$$\frac{|w+z|^p + |w-z|^p}{2} \leq |w|^p + |z|^p \leq \frac{|w+z|^p + |w-z|^p}{2^{p-1}}.$$

(b) Prove that if μ is a measure and $f, g \in \mathcal{L}^p(\mu)$, then

$$\frac{\|f+g\|_p^p + \|f-g\|_p^p}{2} \leq \|f\|_p^p + \|g\|_p^p \leq \frac{\|f+g\|_p^p + \|f-g\|_p^p}{2^{p-1}}.$$

16 Suppose $2 \leq p < \infty$.

(a) Prove that if $w, z \in \mathbf{C}$, then

$$\frac{|w+z|^p + |w-z|^p}{2^{p-1}} \leq |w|^p + |z|^p \leq \frac{|w+z|^p + |w-z|^p}{2}.$$

(b) Prove that if μ is a measure and $f, g \in \mathcal{L}^p(\mu)$, then

$$\frac{\|f+g\|_p^p + \|f-g\|_p^p}{2^{p-1}} \leq \|f\|_p^p + \|g\|_p^p \leq \frac{\|f+g\|_p^p + \|f-g\|_p^p}{2}.$$

[*The inequalities in the two previous exercises are called* Clarkson's Inequalities. *They were discovered by American mathematician James Clarkson in 1936.*]

