Chapter
6

Banach Spaces

We begin this chapter by extending our results on measurable functions and integration to complex-valued functions. This process becomes easy when we apply our previously known results to the real and imaginary parts of complex-valued functions.

Then we rapidly introduce the framework of vector spaces, which will allow us to consider natural collections of measurable functions that are closed under addition and scalar multiplication.

Normed vector spaces and Banach spaces, which are introduced in the second section of this chapter, play a hugely important role in modern analysis. We will see the connections between continuity and boundedness for linear maps on these spaces. We will also see that continuity of a linear map into the scalar field can be determined by considering the set that is mapped to 0.

Market square in Lwów, a city that has been in several countries because of changing international boundaries. Before World War I, Lwów was in Austria-Hungary. During the period between World War I and World War II, Lwów was in Poland. During this time, mathematicians in Lwów, particularly Stefan Banach (1892–1945) and his colleagues, developed the basic results of modern functional analysis. After World War II, Lwów was in USSR. Now Lwów is in Ukraine and is called Lviv.

6A Vector Spaces

Integration of Complex-Valued Functions

Complex numbers were invented so that we can take square roots of negative numbers. The idea is to assume we have a square root of -1, denoted i, that obeys the usual rules of arithmetic. Here are the formal definitions:

6.1 Definition ***complex numbers;*** $\mathbf{C}$

- A *complex number* is an ordered pair (a, b), where $a, b \in \mathbf{R}$, but we will write this as $a + bi$.
- The set of all complex numbers is denoted by $\mathbf{C}$:
$$\mathbf{C} = \{a + bi : a, b \in \mathbf{R}\}.$$
- *Addition and multiplication* on $\mathbf{C}$ are defined by
$$\begin{aligned}(a + bi) + (c + di) &= (a + c) + (b + d)i,\\ (a + bi)(c + di) &= (ac - bd) + (ad + bc)i;\end{aligned}$$
here $a, b, c, d \in \mathbf{R}$.

The symbol i was first used to denote $\sqrt{-1}$ by Swiss mathematician Leonhard Euler in 1777.

If $a \in \mathbf{R}$, then we identify $a + 0i$ with a. Thus we can think of $\mathbf{R}$ as a subset of $\mathbf{C}$. We also usually write $0 + bi$ as just bi, and we usually write $0 + 1i$ as just i. Using multiplication as defined above, you should verify that $i^2 = -1$.

With the definitions as above, $\mathbf{C}$ satisfies the usual rules of arithmetic. Specifically, with addition and multiplication defined as above, $\mathbf{C}$ is a field, as you should verify (see 0.1 in the Appendix for the definition of field). Thus subtraction and division of complex numbers are defined as in any field (see 0.4).

The field $\mathbf{C}$ cannot be made into an ordered field (see Exercise 13 in Section A of the Appendix). However, the useful concept of an absolute value can still be defined on $\mathbf{C}$.

6.2 Definition $\operatorname{Re} z$***;*** $\operatorname{Im} z$***; absolute value; limits***

Suppose $z = a + bi$, where a and b are real numbers.

- The *real part* of z, denoted $\operatorname{Re} z$, is defined by $\operatorname{Re} z = a$.
- The *imaginary part* of z, denoted $\operatorname{Im} z$, is defined by $\operatorname{Im} z = b$.
- The *absolute value* of z, denoted $|z|$, is defined by $|z| = \sqrt{a^2 + b^2}$.
- If $z_1, z_2, \ldots \in \mathbf{C}$ and $L \in \mathbf{C}$, then $\lim\limits_{k\to\infty} z_k = L$ means $\lim\limits_{k\to\infty} |z_k - L| = 0$.

For b a real number, the previous definition of $|b|$ (see 0.9) is consistent with the new definition just given of $|b|$ with b thought of as a complex number. Note that if $z_1, z_2, \ldots$ is a sequence of complex numbers and $L \in \mathbf{C}$, then

$$\lim_{k\to\infty} z_k = L \iff \lim_{k\to\infty} \operatorname{Re} z_k = \operatorname{Re} L \text{ and } \lim_{k\to\infty} \operatorname{Im} z_k = \operatorname{Im} L.$$

We will reduce questions concerning measurability and integration of a complex-valued function to the corresponding questions about the real and imaginary parts of the function. We begin this process with the following definition.

6.3 Definition ***measurable complex-valued function***

Suppose $(X, \mathcal{S})$ is a measurable space. A function $f\colon X \to \mathbf{C}$ is called *$\mathcal{S}$-measurable* if $\operatorname{Re} f$ and $\operatorname{Im} f$ are both $\mathcal{S}$-measurable functions.

See Exercise 4 in this section for two natural conditions that are equivalent to measurability for complex-valued functions.

We will make frequent use of the following result. See Exercise 5 in this section for algebraic combinations of complex-valued measurable functions.

6.4 ***$|f|^p$ is measurable if f is measurable***

Suppose $(X, \mathcal{S})$ is a measurable space, $f\colon X \to \mathbf{C}$ is an $\mathcal{S}$-measurable function, and $0 < p < \infty$. Then $|f|^p$ is an $\mathcal{S}$-measurable function.

Proof The functions $(\operatorname{Re} f)^2$ and $(\operatorname{Im} f)^2$ are $\mathcal{S}$-measurable because the square of an $\mathcal{S}$-measurable function is measurable (by Example 2.44). Thus the function $(\operatorname{Re} f)^2 + (\operatorname{Im} f)^2$ is $\mathcal{S}$-measurable (because the sum of two $\mathcal{S}$-measurable functions is $\mathcal{S}$-measurable by 2.45). Now $\bigl((\operatorname{Re} f)^2 + (\operatorname{Im} f)^2\bigr)^{p/2}$ is $\mathcal{S}$-measurable because it is the composition of a continuous function on $[0, \infty)$ and an $\mathcal{S}$-measurable function (see 2.43 and 2.40). In other words, $|f|^p$ is an $\mathcal{S}$-measurable function. ■

Now we define integration of a complex-valued function by separating the function into its real and imaginary parts.

6.5 Definition ***integral of a complex-valued function***

Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $f\colon X \to \mathbf{C}$ is an $\mathcal{S}$-measurable function such that $\int |f|\, d\mu < \infty$. Then $\int f\, d\mu$ is defined by

$$\int f\, d\mu = \int (\operatorname{Re} f)\, d\mu + i \int (\operatorname{Im} f)\, d\mu.$$

The integral of a complex-valued measurable function is defined above only when the absolute value of the function has a finite integral. In contrast, the integral of every nonnegative measurable function is defined (although the value may equal ∞), and if f is real valued then $\int f\, d\mu$ is defined to be $\int f^+\, d\mu - \int f^-\, d\mu$ if at least one of $\int f^+\, d\mu$ and $\int f^-\, d\mu$ is finite.

You can easily show that if $f, g \colon X \to \mathbf{C}$ are $\mathcal{S}$-measurable functions such that $\int |f| \, d\mu < \infty$ and $\int |g| \, d\mu < \infty$, then

$$\int (f+g) \, d\mu = \int f \, d\mu + \int g \, d\mu.$$

Similarly, the definition of complex multiplication leads to the conclusion that

$$\int \alpha f \, d\mu = \alpha \int f \, d\mu$$

for all $\alpha \in \mathbf{C}$ (see Exercise 7).

The inequality in the result below concerning integration of complex-valued functions does not follow immediately from the corresponding result for real-valued functions. However, the small trick used in the proof below does give a reasonably simple proof.

6.6 ***Bound on the absolute value of an integral***

Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $f \colon X \to \mathbf{C}$ is an $\mathcal{S}$-measurable function such that $\int |f| \, d\mu < \infty$. Then

$$\left| \int f \, d\mu \right| \leq \int |f| \, d\mu.$$

Proof The result clearly holds if $\int f \, d\mu = 0$. Thus assume that $\int f \, d\mu \neq 0$.

Let

$$\alpha = \frac{\left| \int f \, d\mu \right|}{\int f \, d\mu}.$$

Then

$$\begin{aligned}
\left| \int f \, d\mu \right| &= \alpha \int f \, d\mu \\
&= \int \alpha f \, d\mu \\
&= \int \operatorname{Re}(\alpha f) \, d\mu + i \int \operatorname{Im}(\alpha f) \, d\mu \\
&= \int \operatorname{Re}(\alpha f) \, d\mu \\
&\leq \int |\alpha f| \, d\mu \\
&= \int |f| \, d\mu,
\end{aligned}$$

where the second equality holds by Exercise 7, the fourth equality holds because $\left| \int f \, d\mu \right| \in \mathbf{R}$, the inequality on the fifth line holds because $\operatorname{Re} z \leq |z|$ for every complex number z, and the equality in the last line holds because $|\alpha| = 1$. ■

The Bounded Convergence Theorem (3.26) and the Dominated Convergence Theorem (3.30) now clearly hold if the functions $f_1, f_2, \ldots$ and f in the statements of those theorems are allowed to be complex valued.

Vector Spaces and Subspaces

The structure and language of vector spaces will help us focus on certain features of collections of measurable functions. So that we can conveniently make definitions and prove theorems that apply to both real and complex numbers, we adopt the following notation.

6.7 Definition F

From now on, $\mathbf{F}$ stands for either $\mathbf{R}$ or $\mathbf{C}$.

In the definitions that follow, we use f and g to denote elements of V because in the crucial examples the elements of V are functions from a set X to $\mathbf{F}$.

6.8 Definition ***addition, scalar multiplication***

- An *addition* on a set V is a function that assigns an element $f + g \in V$ to each pair of elements $f, g \in V$.
- A *scalar multiplication* on a set V is a function that assigns an element $\alpha f \in V$ to each $\alpha \in \mathbf{F}$ and each $f \in V$.

Now we are ready to give the formal definition of a vector space.

6.9 Definition ***vector space***

A *vector space* (over $\mathbf{F}$) is a set V along with an addition on V and a scalar multiplication on V such that the following properties hold:

commutativity
$f + g = g + f$ for all $f, g \in V$;

associativity
$(f + g) + h = f + (g + h)$ and $(\alpha\beta)f = \alpha(\beta f)$ for all $f, g, h \in V$ and $\alpha, \beta \in \mathbf{F}$;

additive identity
there exists an element $0 \in V$ such that $f + 0 = f$ for all $f \in V$;

additive inverse
for every $f \in V$, there exists $g \in V$ such that $f + g = 0$;

multiplicative identity
$1f = f$ for all $f \in V$;

distributive properties
$\alpha(f + g) = \alpha f + \alpha g$ and $(\alpha + \beta)f = \alpha f + \beta f$ for all $\alpha, \beta \in \mathbf{F}$ and $f, g \in V$.

Most vector spaces that you will encounter are subsets of the vector space $\mathbf{F}^X$ presented in the next example.

6.10 Example *the vector space* $\mathbf{F}^X$

Suppose X is a nonempty set. Let $\mathbf{F}^X$ denote the set of functions from X to $\mathbf{F}$. Addition and scalar multiplication on $\mathbf{F}^X$ are defined as expected: for $f, g \in \mathbf{F}^X$ and $\alpha \in \mathbf{F}$, define

$$(f+g)(x) = f(x) + g(x) \quad \text{and} \quad (\alpha f)(x) = \alpha\bigl(f(x)\bigr)$$

for $x \in X$. Then, as you should verify, $\mathbf{F}^X$ is a vector space; the additive identity in this vector space is the function $0 \in \mathbf{F}^X$ defined by $0(x) = 0$ for all $x \in X$.

6.11 Example $\mathbf{F}^n$*;* $\mathbf{F}^{\mathbf{Z}^+}$

Special case of the previous example: if $n \in \mathbf{Z}^+$ and $X = \{1, \dots, n\}$, then $\mathbf{F}^X$ is the familiar space $\mathbf{R}^n$ or $\mathbf{C}^n$, depending upon whether $\mathbf{F} = \mathbf{R}$ or $\mathbf{F} = \mathbf{C}$.

Another special case: $\mathbf{F}^{\mathbf{Z}^+}$ is the vector space of all sequences of real numbers or complex numbers, again depending upon whether $\mathbf{F} = \mathbf{R}$ or $\mathbf{F} = \mathbf{C}$.

By considering subspaces, we can greatly expand our examples of vector spaces.

6.12 Definition ***subspace***

A subset U of V is called a *subspace* of V if U is also a vector space (using the same addition and scalar multiplication as on V).

The next result gives the easiest way to check whether a subset of a vector space is a subspace.

6.13 ***Conditions for a subspace***

A subset U of V is a subspace of V if and only if U satisfies the following three conditions:

additive identity
$0 \in U$;

closed under addition
$f, g \in U$ implies $f + g \in U$;

closed under scalar multiplication
$\alpha \in \mathbf{F}$ and $f \in U$ implies $\alpha f \in U$.

Proof If U is a subspace of V, then U satisfies the three conditions above by the definition of vector space.

Conversely, suppose U satisfies the three conditions above. The first condition above ensures that the additive identity of V is in U.

The second condition above ensures that addition makes sense on U. The third condition ensures that scalar multiplication makes sense on U.

If $f \in V$, then $0f = (0+0)f = 0f + 0f$. Adding the additive inverse of $0f$ to both sides of this equation shows that $0f = 0$. Now if $f \in U$, then $(-1)f$ is also in U by the third condition above. Because $f + (-1)f = \big(1 + (-1)\big)f = 0f = 0$, we see that $(-1)f$ is an additive inverse of f. Hence every element of U has an additive inverse in U.

The other parts of the definition of a vector space, such as associativity and commutativity, are automatically satisfied for U because they hold on the larger space V. Thus U is a vector space and hence is a subspace of V. ■

The three conditions in 6.13 usually enable us to determine quickly whether a given subset of V is a subspace of V, as illustrated below. All the examples below except for the first bullet point involve concepts from measure theory.

6.14 **Example** *subspaces of* $\mathbf{F}^X$

- The set $C([0,1])$ of continuous real-valued functions on $[0,1]$ is a vector space over $\mathbf{R}$ because the sum of two continuous functions is continuous and a constant multiple of a continuous functions is continuous. In other words, $C([0,1])$ is a subspace of $\mathbf{R}^{[0,1]}$.

- Suppose $(X, \mathcal{S})$ is a measurable space. Then the set of $\mathcal{S}$-measurable functions from X to $\mathbf{F}$ is a subspace of $\mathbf{F}^X$ because the sum of two $\mathcal{S}$-measurable functions is $\mathcal{S}$-measurable and a constant multiple of an $\mathcal{S}$-measurable function is $\mathcal{S}$-measurable.

- Suppose $(X, \mathcal{S}, \mu)$ is a measure space. Then the set $\mathcal{Z}(\mu)$ of $\mathcal{S}$-measurable functions f from X to $\mathbf{F}$ such that $f = 0$ almost everywhere [meaning that $\mu\big(\{x \in X : f(x) \neq 0\}\big) = 0$] is a vector space over $\mathbf{F}$ because the union of two sets with μ-measure 0 is a set with μ-measure 0 [which implies that $\mathcal{Z}(\mu)$ is closed under addition]. Note that $\mathcal{Z}(\mu)$ is a subspace of $\mathbf{F}^X$.

- Suppose $(X, \mathcal{S})$ is a measurable space. Then the set of bounded measurable functions from X to $\mathbf{F}$ is a subspace of $\mathbf{F}^X$ because the sum of two bounded $\mathcal{S}$-measurable functions is a bounded $\mathcal{S}$-measurable function and a constant multiple of a bounded $\mathcal{S}$-measurable function is a bounded $\mathcal{S}$-measurable function.

- Suppose $(X, \mathcal{S}, \mu)$ is a measure space. Then the set of $\mathcal{S}$-measurable functions f from X to $\mathbf{F}$ such that $\int f \, d\mu = 0$ is a subspace of $\mathbf{F}^X$ because of standard properties of integration.

- Suppose $(X, \mathcal{S}, \mu)$ is a measure space. Then the set $\mathcal{L}^1(\mu)$ of $\mathcal{S}$-measurable functions from X to $\mathbf{F}$ such that $\int |f| \, d\mu < \infty$ is a subspace of $\mathbf{F}^X$ (here we have redefined $\mathcal{L}^1(\mu)$ from it previous definition in 3.39 to allow for the possibility that $\mathbf{F} = \mathbf{C}$). The set $\mathcal{L}^1(\mu)$ is closed under addition and scalar multiplication because $\int |f+g| \, d\mu \leq \int |f| \, d\mu + \int |g| \, d\mu$ and $\int |\alpha f| \, d\mu = |\alpha| \int |f| \, d\mu$.

- The set ℓ^1 of all sequences $(a_1, a_2, \dots)$ of elements of $\mathbf{F}$ such that $\sum_{k=1}^{\infty} |a_k| < \infty$ is a subspace of $\mathbf{F}^{\mathbf{Z}^+}$. Note that ℓ^1 is a special case of the example in the previous bullet point (take μ to be counting measure on $\mathbf{Z}^+$).

EXERCISES 6A

1 Suppose $z \in \mathbf{C}$. Prove that

$$\max\{|\operatorname{Re} z|, |\operatorname{Im} z|\} \leq |z| \leq \sqrt{2}\max\{|\operatorname{Re} z|, |\operatorname{Im} z|\}.$$

2 Suppose $z \in \mathbf{C}$. Prove that

$$\frac{|\operatorname{Re} z| + |\operatorname{Im} z|}{\sqrt{2}} \leq |z| \leq |\operatorname{Re} z| + |\operatorname{Im} z|.$$

3 Suppose $w, z \in \mathbf{C}$. Prove that

$$|wz| = |w|\,|z| \quad \text{and} \quad |w + z| \leq |w| + |z|.$$

4 Suppose $(X, \mathcal{S})$ is a measurable space and $f\colon X \to \mathbf{C}$ is a complex-valued function. For conditions (b) and (c) below, identify $\mathbf{C}$ with $\mathbf{R}^2$. Prove that the following are equivalent:

(a) f is $\mathcal{S}$-measurable;
(b) $f^{-1}(E) \in \mathcal{S}$ for every open set E in $\mathbf{R}^2$;
(c) $f^{-1}(E) \in \mathcal{S}$ for every Borel set $E \in \mathcal{B}_2$.

5 Suppose $(X, \mathcal{S})$ is a measurable space and $f, g\colon X \to \mathbf{C}$ are $\mathcal{S}$-measurable. Prove that

(a) $f + g$, $f - g$, and fg are $\mathcal{S}$-measurable functions;
(b) if $g(x) \neq 0$ for all $x \in X$, then $\frac{f}{g}$ is an $\mathcal{S}$-measurable function.

6 Suppose $(X, \mathcal{S})$ is a measurable space and $f_1, f_2, \ldots$ is a sequence of $\mathcal{S}$-measurable functions from X to $\mathbf{C}$. Suppose $\lim_{k\to\infty} f_k(x)$ exists for each $x \in X$. Define $f\colon X \to \mathbf{C}$ by

$$f(x) = \lim_{k\to\infty} f_k(x).$$

Prove that f is an $\mathcal{S}$-measurable function.

7 Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $f\colon X \to \mathbf{C}$ is an $\mathcal{S}$-measurable function such that $\int |f|\, d\mu < \infty$. Prove that if $\alpha \in \mathbf{C}$, then

$$\int \alpha f\, d\mu = \alpha \int f\, d\mu.$$

8 Suppose V is a vector space. Show that the intersection of every collection of subspaces of V is a subspace of V.

9 Suppose V and W are vector spaces. Define $V \times W$ by

$$V \times W = \{(f, g) : f \in V \text{ and } g \in W\}.$$

Define addition and scalar multiplication on $V \times W$ by

$$(f_1, g_1) + (f_2, g_2) = (f_1 + f_2, g_1 + g_2) \quad \text{and} \quad \alpha(f, g) = (\alpha f, \alpha g).$$

Prove that $V \times W$ is a vector space with these operations.

6B Normed Vector Spaces

Norms and Cauchy Sequences

This section begins with a crucial definition.

6.15 Definition *norm; normed vector space*

A *norm* on a vector space V (over $\mathbf{F}$) is a function $\|\cdot\|: V \to [0, \infty)$ such that

- $\|f\| = 0$ if and only if $f = 0$ (**positive definite**);
- $\|\alpha f\| = |\alpha| \, \|f\|$ for all $\alpha \in \mathbf{F}$ and $f \in V$ (**homogeneity**);
- $\|f + g\| \leq \|f\| + \|g\|$ for all $f, g \in V$ (**triangle inequality**).

A *normed vector space* is a vector space V along with a norm on V.

6.16 Example *normed vector spaces*

- Suppose $n \in \mathbf{Z}^+$. Define $\|\cdot\|_1$ and $\|\cdot\|_\infty$ on $\mathbf{F}^n$ by
$$\|(a_1, \ldots, a_n)\|_1 = |a_1| + \cdots + |a_n|$$
and
$$\|(a_1, \ldots, a_n)\|_\infty = \max\{|a_1|, \ldots, |a_n|\}.$$
Then $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are norms on $\mathbf{F}^n$, as you should verify.

- On ℓ^1 (see the last bullet point in Example 6.14 for the definition of ℓ^1), define $\|\cdot\|_1$ by
$$\|(a_1, a_2, \ldots)\|_1 = \sum_{k=1}^{\infty} |a_k|.$$
Then $\|\cdot\|_1$ a norm on ℓ^1, as you should verify.

- Suppose X is a nonempty set and $b(X)$ is the subspace of $\mathbf{F}^X$ consisting of the bounded functions from X to $\mathbf{F}$. For f a bounded function from X to $\mathbf{F}$, define $\|f\|$ by
$$\|f\| = \sup\{|f(x)| : x \in X\}.$$
Then $\|\cdot\|$ is a norm on $b(X)$, as you should verify.

- Let $C([0,1])$ denote the vector space of continuous functions from the interval $[0,1]$ to $\mathbf{F}$. Define $\|\cdot\|$ on $C([0,1])$ by
$$\|f\| = \int_0^1 |f|.$$
Then $\|\cdot\|$ is a norm on $C([0,1])$, as you should verify.

Sometimes examples that do not satisfy a definition help you gain understanding.

6.17 Example *not norms*

- Let $\mathcal{L}^1(\lambda)$ denote the vector space of Borel (or Lebesgue) measurable functions $f\colon \mathbf{R} \to \mathbf{F}$ such that $\int |f|\, d\lambda < \infty$, where λ is Lebesgue measure on $\mathbf{R}$. Define $\|\cdot\|_1$ on $\mathcal{L}^1(\lambda)$ by
$$\|f\|_1 = \int |f|\, d\lambda.$$
Then $\|\cdot\|_1$ satisfies the homogeneity condition and the triangle inequality on $\mathcal{L}^1(\lambda)$, as you should verify. However, $\|\cdot\|_1$ is not a norm on $\mathcal{L}^1(\lambda)$ because the positive definite condition is not satisfied. Specifically, if E is a nonempty Borel subset of $\mathbf{R}$ with Lebesgue measure 0 (for example, E might consist of a single point in $\mathbf{R}$), then $\|\chi_E\|_1 = 0$ but $\chi_E \neq 0$. In the next chapter, we will discuss a modification of $\mathcal{L}^1(\lambda)$ that removes this problem.

- If $n \in \mathbf{Z}^+$ and $\|\cdot\|$ is defined on $\mathbf{F}^n$ by
$$\|(a_1, \dots, a_n)\| = |a_1|^{1/2} + \cdots + |a_n|^{1/2},$$
then $\|\cdot\|$ satisfies the positive definite condition and the triangle inequality (as you should verify). However, $\|\cdot\|$ as defined above is not a norm because it does not satisfy the homogeneity condition.

- If $\|\cdot\|_{1/2}$ is defined on $\mathbf{F}^n$ by
$$\|(a_1, \dots, a_n)\|_{1/2} = \bigl(|a_1|^{1/2} + \cdots + |a_n|^{1/2}\bigr)^2,$$
then $\|\cdot\|_{1/2}$ satisfies the positive definite condition and the homogeneity condition. However, if $n > 1$ then $\|\cdot\|_{1/2}$ is not a norm on $\mathbf{F}^n$ because the triangle inequality is not satisfied (as you should verify).

The concept of norm allows us to extend many familiar notions from elementary analysis on $\mathbf{R}$ to the context of normed vector spaces, often using the same definitions except that the absolute value is replaced by the norm.

6.18 Definition ***limit in*** V

Suppose $f_1, f_2, \dots$ is a sequence in a normed vector space V and $f \in V$. Then $\lim\limits_{k\to\infty} f_k = f$ means $\lim\limits_{k\to\infty} \|f_k - f\| = 0$.

In other words, a sequence $f_1, f_2, \dots$ in V converges to $f \in V$ if for every $\varepsilon > 0$, there exists $n \in \mathbf{Z}^+$ such that $\|f_k - f\| < \varepsilon$ for all integers $k \geq n$.

The next definition is useful for showing (in some normed vector spaces) that a sequence has a limit, even when we do not have a good candidate for that limit.

6.19 Definition ***Cauchy sequence***

A sequence $f_1, f_2, \dots$ in a normed vector space V is called a *Cauchy sequence* if for every $\varepsilon > 0$, there exists $n \in \mathbf{Z}^+$ such that $\|f_j - f_k\| < \varepsilon$ for all integers $j \geq n$ and $k \geq n$.

Entrance to the École Polytechnique (Paris), where Augustin-Louis Cauchy (1789–1857) was a student and a faculty member. Cauchy wrote almost 800 mathematics papers, in addition his highly influential textbook Cours d'Analyse *(published in 1821), which greatly influenced the development of analysis.*

Every sequence in a normed vector space that has a limit is a Cauchy sequence, as you are asked to show in Exercise 3. Normed vector spaces that satisfy the converse have a special name.

6.20 Definition ***Banach space***

A normed vector space V is called a *Banach space* if every Cauchy sequence in V converges to some element of V.

The verifications of the assertions in Examples 6.21 and 6.22 are left to the reader as exercises.

6.21 Example *Banach spaces*

- The vector space $C([0,1])$ with the norm defined by $\|f\| = \sup_{x\in[0,1]}|f(x)|$ is a Banach space.
- The vector space ℓ^1 with the norm defined by $\|(a_1, a_2, \dots)\|_1 = \sum_{k=1}^{\infty}|a_k|$ is a Banach space.

6.22 Example *not a Banach space*

- The vector space $C([0,1])$ with the norm defined by $\|f\| = \int_0^1|f|$ is not a Banach space.
- The vector space ℓ^1 with the norm defined by $\|(a_1, a_2, \dots)\|_\infty = \sup_{k\in\mathbf{Z}^+}|a_k|$ is not a Banach space.

6.23 Definition ***infinite sum in a normed vector space***

Suppose $g_1, g_2, \ldots$ is a sequence in a normed vector space V. Then $\sum_{k=1}^{\infty} g_k$ is defined by

$$\sum_{k=1}^{\infty} g_k = \lim_{n\to\infty} \sum_{k=1}^{n} g_k$$

if this limit exists, in which case the infinite series is said to *converge*.

Recall from your calculus course that if $a_1, a_2, \ldots$ is a sequence of real numbers such that $\sum_{k=1}^{\infty} |a_k| < \infty$, then $\sum_{k=1}^{\infty} a_k$ converges. The next result states that the analogous property for normed vector spaces characterizes Banach spaces.

6.24 $\left(\sum_{k=1}^{\infty} \|g_k\| < \infty \implies \sum_{k=1}^{\infty} g_k \textbf{ converges}\right) \iff$ ***Banach space***

Suppose V is a normed vector space. Then V is a Banach space if and only if $\sum_{k=1}^{\infty} g_k$ converges for every sequence $g_1, g_2, \ldots$ in V such that $\sum_{k=1}^{\infty} \|g_k\| < \infty$.

Proof First suppose V is a Banach space. Suppose $g_1, g_2, \ldots$ is a sequence in V such that $\sum_{k=1}^{\infty} \|g_k\| < \infty$. Suppose $\varepsilon > 0$. Let $n \in \mathbf{Z}^+$ be such that $\sum_{m=n}^{\infty} \|g_m\| < \varepsilon$. For $j \in \mathbf{Z}^+$, let f_j denote the partial sum defined by

$$f_j = g_1 + \cdots + g_j.$$

If $j > k \geq n$, then

$$\begin{aligned} \|f_j - f_k\| &= \|g_{k+1} + \cdots + g_j\| \\ &\leq \|g_{k+1}\| + \cdots + \|g_j\| \\ &\leq \sum_{m=n}^{\infty} \|g_m\| \\ &< \varepsilon. \end{aligned}$$

Thus $f_1, f_2, \ldots$ is a Cauchy sequence in V. Because V is a Banach space, we conclude that $f_1, f_2, \ldots$ converges to some element of V, which is precisely what it means for $\sum_{k=1}^{\infty} g_k$ to converge, completing one direction of the proof.

To prove the other direction, suppose $\sum_{k=1}^{\infty} g_k$ converges for every sequence $g_1, g_2, \ldots$ in V such that $\sum_{k=1}^{\infty} \|g_k\| < \infty$. Suppose $f_1, f_2, \ldots$ is a Cauchy sequence in V. We want to prove that $f_1, f_2, \ldots$ converges to some element of V. It suffices to show that some subsequence of $f_1, f_2, \ldots$ converges (by Exercise 11). Dropping to a subsequence (but not relabelling) and setting $f_0 = 0$, we can assume that

$$\sum_{k=1}^{\infty} \|f_k - f_{k-1}\| < \infty.$$

Hence $\sum_{k=1}^{\infty} (f_k - f_{k-1})$ converges. The partial sum of this series after n terms equals f_n. Thus $\lim_{n\to\infty} f_n$ exists, completing the proof. ■

Open Sets, Closed Sets, and Continuity

In future chapters, we will need to use a normed vector space's topological features, which we introduce now. The material in this subsection will be easy to understand if you replace familiar definitions and proofs in the context of $\mathbf{R}$ or $\mathbf{R}^n$ (see the Appendix) with the corresponding notions for a normed vector space (with the absolute value on $\mathbf{R}$ replaced by the norm on a normed vector space). Thus in this subsection more details than usual are left to the reader to verify; verifying those details is the best way to gain good understanding of this material.

6.25 Definition ***open ball;*** $B(f,r)$

Suppose V is a normed vector space, $f \in V$, and $r > 0$.

- The *open ball* centered at f with radius r is denoted $B(f,r)$ and is defined by
$$B(f,r) = \{g \in V : \|g - f\| < r\}.$$
- The *closed ball* centered at f with radius r is denoted $\overline{B}(f,r)$ and is defined by
$$\overline{B}(f,r) = \{g \in V : \|g - f\| \leq r\}.$$

Now we define a subset of a normed vector space to be open if every point in the set is the center of an open ball that is contained in the set.

6.26 Definition ***open set***

A subset E of a normed vector space V is called *open* if for every $f \in E$, there exists $r > 0$ such that $B(f,r) \subset E$.

6.27 Example *open balls are open*

Every open ball of a normed vector space V is open. To see this, suppose $f \in V$ and $r > 0$. To show that $B(f,r)$ is open, suppose that $g \in B(f,r)$. If $h \in B(g, r - \|g - f\|)$, then

$$\|h - f\| \leq \|h - g\| + \|g - f\| < (r - \|g - f\|) + \|g - f\| = r,$$

which implies that $h \in B(f,r)$. Thus $B(g, r - \|g - f\|) \subset B(f,r)$, which implies that $B(f,r)$ is open.

Closed sets are defined in terms of open sets.

6.28 Definition ***closed subset***

A subset of a normed vector space V is called *closed* if its complement in V is open.

6.29 Example *closed balls are closed*

Every closed ball of a normed vector space V is closed. To see this, suppose $f \in V$ and $r > 0$. To show that $\overline{B}(f,r)$ is closed, we must show that $V \setminus \overline{B}(f,r)$ is open. To do this, suppose $g \in V \setminus \overline{B}(f,r)$. If $h \in B(g, \|g - f\| - r)$, then

$$\|h - f\| \geq \|g - f\| - \|g - h\| > \|g - f\| - (\|g - f\| - r) = r,$$

which implies that $h \in V \setminus \overline{B}(f,r)$. Thus $B(g, \|g - f\| - r) \subset V \setminus \overline{B}(f,r)$, which implies that $V \setminus \overline{B}(f,r)$ is open. Thus $\overline{B}(f,r)$ is closed.

Now we define the closure of a subset of a normed vector space.

6.30 Definition ***closure***

Suppose V is a normed vector space and $E \subset V$. The *closure* of E, denoted $\overline{E}$, is defined by

$$\overline{E} = \{g \in V : \text{ for every } \varepsilon > 0, \text{ there exists } f \in E \text{ such that } \|g - f\| < \varepsilon\}.$$

In other words, the closure of E is the set of $g \in V$ such that every open ball centered at g contains at least one element of E.

6.31 Example *closure of an open ball*

Suppose V is a normed vector space. Then the closure of each open ball is the corresponding closed ball. In other words, if $f \in V$ and $r > 0$ then

$$\overline{B(f,r)} = \overline{B}(f,r),$$

as you should verify.

The proof of the next result is left as an exercise that provides good practice in using these concepts. If you want a hint, look at the proof of 0.62 in the Appendix, which may give a helpful pattern for the proof of part of the following result.

6.32 ***Closure***

Suppose V is a normed vector space and $E \subset V$. Then

(a) $\overline{E} = \{g \in V : \text{ there exist } f_1, f_2, \dots \text{ in } E \text{ such that } \lim\limits_{k\to\infty} f_k = g\}$;

(b) $\overline{E}$ equals the intersection of all closed subsets of V that contain E;

(c) $\overline{E}$ is a closed subset of V;

(d) E is closed if and only if $\overline{E} = E$;

(e) E is closed if and only if E contains the limit of every convergent sequence of elements of E.

The definition of continuity that follows uses the same pattern as the definition for a function from a subset of $\mathbf{R}^m$ to $\mathbf{R}^n$ (see 0.75 in the Appendix).

6.33 Definition ***continuity***

Suppose V and W are normed vector spaces and $T \colon V \to W$ is a function.

- For $g \in V$, the function T is called *continuous* at g if for every $\varepsilon > 0$, there exists $\delta > 0$ such that
$$\|T(f) - T(g)\| < \varepsilon$$
for all $f \in V$ with $\|f - g\| < \delta$.
- The function T is called *continuous* if T is continuous at g for every $g \in V$.

The next result gives equivalent conditions for continuity. Recall that $T^{-1}(E)$ is called the inverse image of E and is defined to be $\{f \in V : T(f) \in E\}$. Thus the equivalence of the (a) and (c) below could be restated as saying that a function is continuous if and only if the inverse image of every open set is open. The equivalence of the (a) and (d) below could be restated as saying that a function is continuous if and only if the inverse image of every closed set is closed.

6.34 ***Equivalent conditions for continuity***

Suppose V and W are normed vector spaces and $T \colon V \to W$ is a function. Then the following are equivalent:

(a) T is continuous.

(b) $\lim\limits_{k\to\infty} f_k = f$ in V implies $\lim\limits_{k\to\infty} T(f_k) = T(f)$ in W.

(c) $T^{-1}(E)$ is an open subset of V for every open set $E \subset W$.

(d) $T^{-1}(E)$ is a closed subset of V for every closed set $E \subset W$.

Proof We will prove that (b) implies (d). Thus suppose (b) holds. Suppose E is a closed subset of W. We need to prove that $T^{-1}(E)$ is closed. To do this, suppose $f_1, f_2, \ldots$ is a sequence in $T^{-1}(E)$ and $\lim_{k\to\infty} f_k = f$ for some $f \in V$. Because (b) holds, we know that $\lim_{k\to\infty} T(f_k) = T(f)$. Because each $f_k \in T^{-1}(E)$ for each $k \in \mathbf{Z}^+$, we know that $T(f_k) \in E$ for each $k \in \mathbf{Z}^+$. Because E is closed, this implies that $T(f) \in E$. Thus $f \in T^{-1}(E)$, which implies that $T^{-1}(E)$ is closed [by 6.32(e)], completing the proof that (b) implies (d).

The proof that (c) and (d) are equivalent follows from the equation
$$T^{-1}(W \setminus E) = V \setminus T^{-1}(E)$$
for every $E \subset W$ and the fact that a set is open if and only if its complement (in the appropriate normed vector space) is closed.

The proof of the remaining parts of this result are left as an exercise that should help strengthen understanding of these concepts. ■

Bounded Linear Maps

When dealing with two or more vector spaces, as in the definition below, assume that the vector spaces are over the same field (either **R** or **C**, but denoted in this book as **F** to give us the flexibility to consider both cases).

The notation Tf, in addition to the standard functional notation $T(f)$, is often used when considering linear maps, which we now define.

6.35 Definition ***linear map***

Suppose V and W are vector spaces. A function $T\colon V \to W$ is called *linear* if

- $T(f+g) = Tf + Tg$ for all $f, g \in V$;
- $T(\alpha f) = \alpha Tf$ for all $\alpha \in \mathbf{F}$ and $f \in V$.

A linear function is often called a *linear map*.

The set of linear maps from a vector space V to a vector space W is itself a vector space, using the usual operations of addition and scalar multiplication of functions. Most attention in analysis is focused on the subspace of bounded linear functions, defined below.

In the next definition, we have two normed vector spaces, V and W, which may have different norms. However, we use the same notation $\|\cdot\|$ for both norms (and for the norm of a linear map from V to W) because the context makes the meaning clear. For example, in the definition below, f is in V and thus $\|f\|$ refers to the norm in V. Similarly, $Tf \in W$ and thus $\|Tf\|$ refers to the norm in W.

6.36 Definition ***bounded linear map;*** $\|T\|$***;*** $B(V,W)$

Suppose V and W are normed vector spaces and $T\colon V \to W$ is a linear map.

- The norm of T, denoted $\|T\|$, is defined by
$$\|T\| = \sup\{\|Tf\| : f \in V \text{ and } \|f\| \le 1\}.$$
- T is called *bounded* if $\|T\| < \infty$.
- The set of bounded linear maps from V to W is denoted $B(V,W)$.

6.37 Example *bounded linear map*

Let $C([0,3])$ be the normed vector space of continuous functions from $[0,3]$ to **F**, with $\|f\| = \sup_{x\in[0,3]}|f(x)|$. Define $T\colon C([0,3]) \to C([0,3])$ by
$$(Tf)(x) = x^2 f(x).$$
Then T is a bounded linear map and $\|T\| = 9$, as you should verify.

6.38 Example *linear map that is not bounded*

Let V be the normed vector space of sequences $(a_1, a_2, \dots)$ of elements of $\mathbf{F}$ such that $a_k = 0$ for all but finitely many $k \in \mathbf{Z}^+$, with $\|(a_1, a_2, \dots)\|_\infty = \max_{k \in \mathbf{Z}^+} |a_k|$. On ℓ^1, use the norm defined by $\|(a_1, a_2, \dots)\|_1 = \sum_{k=1}^\infty |a_k|$. Define $T \colon V \to \ell^1$ by

$$T(a_1, a_2, a_3, \dots) = (a_1, 2a_2, 3a_3, \dots).$$

Then T is a linear map that is not bounded, as you should verify.

The next result shows if V and W are normed vector spaces, then $B(V, W)$ is a normed vector space with the norm defined above.

6.39 $\|\cdot\|$ ***is a norm on*** $B(V, W)$

Suppose V and W are normed vector spaces. Then the function $\|\cdot\|$ is a norm on $B(V, W)$.

Proof Suppose $S, T \in B(V, W)$. then

$$\begin{aligned}
\|S + T\| &= \sup\{\|(S + T)f\| : f \in V \text{ and } \|f\| \le 1\} \\
&\le \sup\{\|Sf\| + \|Tf\| : f \in V \text{ and } \|f\| \le 1\} \\
&\le \sup\{\|Sf\| : f \in V \text{ and } \|f\| \le 1\} \\
&\qquad\qquad + \sup\{\|Tf\| : f \in V \text{ and } \|f\| \le 1\} \\
&= \|S\| + \|T\|.
\end{aligned}$$

The inequality above shows that $\|\cdot\|$ satisfies the triangle inequality on $B(V, W)$. The verification of the other properties required for a normed vector space is left to the reader. ∎

Be sure that you are comfortable using all four equivalent formulas for $\|T\|$ shown in Exercise 26. For example, you should often think of $\|T\|$ as the smallest number such that $\|Tf\| \le \|T\| \, \|f\|$ for all f in the domain of T.

Note that in the next result, the hypothesis requires that W be a Banach space but there is no requirement for V to be a Banach space.

6.40 $B(V, W)$ ***is a Banach space if*** W ***is a Banach space***

Suppose V is a normed vector space and W is a Banach space. Then $B(V, W)$ is a Banach space.

Proof Suppose $T_1, T_2, \dots$ is a Cauchy sequence in $B(V, W)$. If $f \in V$, then

$$\|T_j f - T_k f\| \le \|T_j - T_k\| \, \|f\|,$$

which implies that $T_1 f, T_2 f, \dots$ is a Cauchy sequence in W. Because W is a Banach space, this implies that $T_1 f, T_2 f, \dots$ has a limit in W, which we call Tf.

We have now defined a function $T\colon V \to W$. The reader should verify that T is a linear map. Clearly

$$\begin{aligned}\|Tf\| &\leq \sup\{\|T_k f\| : k \in \mathbf{Z}^+\} \\ &\leq \bigl(\sup\{\|T_k\| : k \in \mathbf{Z}^+\}\bigr)\|f\|\end{aligned}$$

for each $f \in V$. The last supremum above is finite because every Cauchy sequence is bounded (see Exercise 4). Thus $T \in \mathcal{B}(V, W)$.

We still need to show that $\lim_{k\to\infty}\|T_k - T\| = 0$. To do this, suppose $\varepsilon > 0$. Let $n \in \mathbf{Z}^+$ be such that $\|T_j - T_k\| < \varepsilon$ for all $j \geq n$ and $k \geq n$. Suppose $j \geq n$ and suppose $f \in V$. Then

$$\begin{aligned}\|(T_j - T)f\| &= \lim_{k\to\infty}\|T_j f - T_k f\| \\ &\leq \varepsilon\|f\|.\end{aligned}$$

Thus $\|T_j - T\| \leq \varepsilon$, completing the proof. ■

The next result shows that the phrase *bounded linear map* means the same as the phrase *continuous linear map*.

6.41 *Continuity is equivalent to boundedness for linear maps*

A linear map from one normed vector space to another normed vector space is continuous if and only if it is bounded.

Proof Suppose V and W are normed vector spaces and $T\colon V \to W$ is linear.

First suppose T is not bounded. Thus there exists a sequence $f_1, f_2, \ldots$ in V such that $\|f_k\| \leq 1$ for each $k \in \mathbf{Z}^+$ and $\|Tf_k\| \to \infty$ as $k \to \infty$. Hence

$$\lim_{k\to\infty}\frac{f_k}{\|Tf_k\|} = 0 \quad \text{and} \quad T\Bigl(\frac{f_k}{\|Tf_k\|}\Bigr) = \frac{Tf_k}{\|Tf_k\|} \not\to 0,$$

where the nonconvergence to 0 holds because $Tf_k/\|Tf_k\|$ has norm 1 for every $k \in \mathbf{Z}^+$. The displayed line above implies that T is not continuous, completing the proof in one direction.

To prove the other direction, now suppose that T is bounded. Suppose $f \in V$ and $f_1, f_2, \ldots$ is a sequence in V such that $\lim_{k\to\infty} f_k = f$. Then

$$\begin{aligned}\|Tf_k - Tf\| &= \|T(f_k - f)\| \\ &\leq \|T\|\,\|f_k - f\|.\end{aligned}$$

Thus $\lim_{k\to\infty} Tf_k = Tf$. Hence T is continuous, completing the proof in the other direction. ■

Exercise 28 gives several additional equivalent conditions for a linear map to be continuous.

Linear Functionals

Linear maps into the scalar field $\mathbf{F}$ are so important that they get a special name.

6.42 Definition ***linear functional***

A *linear functional* on a vector space V is a linear map from V to $\mathbf{F}$.

When we think of the scalar field $\mathbf{F}$ as a normed vector space, as in the next example, the norm $\|z\|$ of a number $z \in \mathbf{F}$ is always intended to be just the usual absolute value $|z|$. This norm makes $\mathbf{F}$ a Banach space.

6.43 Example *linear functional*

Let V be the normed vector space of sequences $(a_1, a_2, \ldots)$ of elements of $\mathbf{F}$ such that $a_k = 0$ for all but finitely many $k \in \mathbf{Z}^+$. Define $\varphi \colon V \to \mathbf{F}$ by

$$\varphi(a_1, a_2, \ldots) = \sum_{k=1}^{\infty} a_k.$$

Then φ is a linear functional on V.

- If we make V a normed vector space with the norm $\|(a_1, a_2, \ldots)\|_1 = \sum_{k=1}^{\infty} |a_k|$, then φ is a bounded linear functional on V, as you should verify.
- If we make V a normed vector space with the norm $\|(a_1, a_2, \ldots)\|_\infty = \max_{k \in \mathbf{Z}^+} |a_k|$, then φ is not a bounded linear functional on V, as you should verify.

6.44 Definition ***null space;*** null T

Suppose V and W are vector spaces and $T \colon V \to W$ is a linear map. Then the *null space* of T is denoted by null T and is defined by

$$\operatorname{null} T = \{f \in V : Tf = 0\}.$$

If T is a linear map on a vector space V, then null T is a subspace of V, as you should verify. If T is a continuous linear map from a normed vector space V to a normed vector space W, then null T is a closed subspace of V because $\operatorname{null} T = T^{-1}(\{0\})$ and the inverse image of the closed set $\{0\}$ is closed [by 6.34(d)].

*The term **kernel** is also used in the mathematics literature with the same meaning as **null space**. This book uses **null space** instead of **kernel** because **null space** better captures the connection with 0. Also, **kernel** has another completely different meaning in the theory of integral operators, as we will see later.*

The converse of the last sentence above is not true, in the sense that a linear map between normed vector spaces might have a closed null space but not be continuous. For example, the linear map in Example 6.38 has a closed null space (equal to $\{0\}$) but it is not continuous.

However, the next result states that for linear functionals (as opposed to more general linear maps) having a closed null space is equivalent to continuity.

6.45 *Bounded linear functionals*

Suppose V is a normed vector space and $\varphi\colon V \to \mathbf{F}$ is a linear functional that is not identically 0. Then the following are equivalent:

(a) φ is a bounded linear functional.

(b) φ is a continuous linear functional.

(c) $\operatorname{null} \varphi$ is a closed subspace of V.

(d) $\overline{\operatorname{null} \varphi} \neq V$.

Proof The equivalence of (a) and (b) is just a special case of 6.41.

To prove that (b) implies (c), suppose that φ is a continuous linear functional. Then $\operatorname{null} \varphi$, which is the inverse image of the closed set $\{0\}$, is a closed subset of V by 6.34(d). Thus (b) implies (c).

To prove that (c) implies (a), we will show that the negation of (a) implies the negation of (c). Thus suppose φ is not bounded. Thus there is a sequence $f_1, f_2, \dots$ in V such that $\|f_k\| \leq 1$ and $|\varphi(f_k)| \geq k$ for each $k \in \mathbf{Z}^+$. Now

6.46 $$\frac{f_1}{\varphi(f_1)} - \frac{f_k}{\varphi(f_k)} \in \operatorname{null} \varphi \text{ for each } k \in \mathbf{Z}^+$$

and

6.47 $$\lim_{k\to\infty} \Bigl(\frac{f_1}{\varphi(f_1)} - \frac{f_k}{\varphi(f_k)}\Bigr) = \frac{f_1}{\varphi(f_1)}.$$

Clearly

$$\varphi\Bigl(\frac{f_1}{\varphi(f_1)}\Bigr) = 1 \text{ and thus } \frac{f_1}{\varphi(f_1)} \notin \operatorname{null} \varphi.$$

Hence 6.46 and 6.47 imply that $\operatorname{null} \varphi$ is not closed, completing the proof that the negation of (a) implies the negation of (c). Thus (c) implies (a).

At this stage of the proof, we now know that (a), (b), and (c) are equivalent to each other.

Using the hypothesis that φ is not identically 0, we see that (c) implies (d). To complete the proof, we need only show that (d) implies (c), which we will do by showing that the negation of (c) implies the negation of (d). Thus suppose $\operatorname{null} \varphi$ is not a closed subspace of V. Because $\operatorname{null} \varphi$ is a subspace of V, we know that $\overline{\operatorname{null} \varphi}$ is also a subspace of V (see Exercise 20). Let $f \in \overline{\operatorname{null} \varphi} \setminus \operatorname{null} \varphi$. Suppose $g \in V$. Then

$$g = \Bigl(g - \frac{\varphi(g)}{\varphi(f)} f\Bigr) + \frac{\varphi(g)}{\varphi(f)} f.$$

The term in large parentheses above is in $\operatorname{null} \varphi$ and hence is in $\overline{\operatorname{null} \varphi}$. The term above following the plus sign is a scalar multiple of f and thus is in $\overline{\operatorname{null} \varphi}$. Because the equation above writes g as the sum of two elements of $\overline{\operatorname{null} \varphi}$, we conclude that $g \in \overline{\operatorname{null} \varphi}$. Hence we have shown that $V = \overline{\operatorname{null} \varphi}$, completing the proof that the negation of (c) implies the negation of (d). ∎

The proof of the previous result makes major use of dividing by expressions of the form $\varphi(f)$, which would not make sense for a linear mapping into a vector space other than $\mathbf{F}$. However, see Exercise 32 for a similar result for linear mappings into $\mathbf{F}^n$.

The existence of discontinuous linear functionals (as shown, for example, in Example 6.43) means that we need to readjust our intuition from the situation on $\mathbf{F}^n$ where all linear functionals are continuous (see Exercise 33).

EXERCISES 6B

1 Show that the map $f \mapsto \|f\|$ from a normed vector space V to $\mathbf{F}$ is continous (where the norm on $\mathbf{F}$ is the usual absolute value).

2 Show that the functions defined in the last two bullet points of Example 6.17 are not norms.

3 Prove that every sequence in a normed vector space that has a limit is a Cauchy sequence.

4 Prove that each Cauchy sequence in a normed vector space is bounded (meaning that there is a real number that is greater than the norm of every element in the Cauchy sequence).

5 Show that if $n \in \mathbf{Z}^+$, then $\mathbf{F}^n$ is a Banach space with both the norms used in the first bullet point of Example 6.16.

6 Suppose X is a nonempty set and $b(X)$ is the vector space of bounded functions from X to $\mathbf{F}$. Prove that if $\|\cdot\|$ is defined on $b(X)$ by $\|f\| = \sup_{x \in X} |f(x)|$, then $b(X)$ is a Banach space.

7 Show that ℓ^1 with the norm defined by $\|(a_1, a_2, \ldots)\|_\infty = \sup_{k \in \mathbf{Z}^+} |a_k|$ is not a Banach space.

8 Show that ℓ^1 with the norm defined by $\|(a_1, a_2, \ldots)\|_1 = \sum_{k=1}^\infty |a_k|$ is a Banach space.

9 Show that the vector space $C([0,1])$ of continuous functions from $[0,1]$ to $\mathbf{F}$ with the norm defined by $\|f\| = \sup_{x \in [0,1]} |f(x)|$ is a Banach space.

10 Show that $C([0,1])$ with the norm defined by $\|f\| = \int_0^1 |f|$ is not a Banach space.

11 Suppose $f_1, f_2, \ldots$ is a Cauchy sequence in a normed vector space V. Prove that if some subsequence of $f_1, f_2, \ldots$ converges to some element of V, then $f_1, f_2, \ldots$ converges.

12 Suppose U is a subspace of a normed vector space V such that some open ball of V is contained in U. Prove that $U = V$.

13 Suppose U is a subspace of a Banach space V. Prove that U is a Banach space (in the norm that it inherits from V) if and only if U is a closed subspace of V.

14 Suppose V is a normed vector space.

(a) Prove that the union of every collection of open subsets of V is an open subset of V.

(b) Prove that the intersection of every finite collection of open subsets of V is an open subset of V.

15 Suppose V is a normed vector space.

(a) Prove that the intersection of every collection of closed subsets of V is a closed subset of V.

(b) Prove that the union of every finite collection of closed subsets of V is a closed subset of V.

16 Prove that every open subset of a normed vector space V is the union of some sequence of closed subsets of V.

17 Prove that the only subsets of a normed vector space V that are both open and closed are $\varnothing$ and V.

18 Prove the assertion in Example 6.31 that if V is a normed vector space, $f \in V$, and $r > 0$, then

$$\overline{B(f,r)} = \overline{B}(f,r).$$

19 Prove 6.32.

20 Prove that if V is a normed vector space and U is a subspace of V, then $\overline{U}$ is a subspace of V.

21 Suppose that U is a normed vector space.

(a) Show that the set W of all Cauchy sequences of elements of U is a vector space under natural operations of addition and scalar multiplication.

(b) For $F = (f_1, f_2, \ldots)$ and $G = (g_1, g_2, \ldots)$ in W, define $F \equiv G$ to mean that

$$\lim_{k \to \infty} \|f_k - g_k\| = 0.$$

Show that $\equiv$ is an equivalence relation on W.

(c) Show that the set V of equivalence classes of elements of W, under the equivalence relation defined in (b), is a vector space under natural operations of addition and scalar multiplication.

(d) Show that there is a natural way to make V into a normed vector space and that with this norm, V is a Banach space.

(e) Show that the map from U to V that takes $f \in U$ to the equivalence class of $(f, f, f, \ldots)$ is an isometry of U into V.

(f) Explain why (e) shows that every normed vector space is a subspace of some Banach space.

22 Suppose U is a subspace of a normed vector space V. Suppose also that W is a Banach space and $S\colon U \to W$ is a bounded linear map. Prove that there exists a bounded linear map $T\colon \overline{U} \to W$ such that

$$T|_U = S \quad \text{and} \quad \|T\| = \|S\|.$$

23 Give an example to show that the previous exercise can fail if the hypothesis that W is a Banach space is replaced with the hypothesis that W is a normed vector space.

24 Prove the parts of 6.34 that were not proved in the text.

25 For readers familiar with the quotient of a vector space and a subspace: Suppose V is a normed vector space and U is a subspace of V. Define $\|\cdot\|$ on V/U by

$$\|f + U\| = \inf\{\|f + g\| : g \in U\}.$$

(a) Prove that $\|\cdot\|$ is a norm on V/U if and only if U is a closed subspace of V.

(b) Prove that if V is a Banach space and U is a closed subspace of V, then V/U (with the norm defined above) is a Banach space.

(c) Prove that if U is a Banach space (with the norm it inherits from V) and V/U is a Banach space (with the norm defined above), then V is a Banach space.

26 Suppose V and W are normed vector spaces with $V \neq \{0\}$ and $T\colon V \to W$ is a linear map.

(a) Show that $\|T\| = \sup\{\|Tf\| : f \in V \text{ and } \|f\| < 1\}$.

(b) Show that $\|T\| = \sup\{\|Tf\| : f \in V \text{ and } \|f\| = 1\}$.

(c) Show that $\|T\| = \inf\{c \in [0, \infty) : \|Tf\| \leq c\|f\| \text{ for all } f \in V\}$.

(d) Show that $\|T\| = \sup\Big\{\dfrac{\|Tf\|}{\|f\|} : f \in V \text{ and } f \neq 0\Big\}$.

27 Suppose U, V, and W are normed vector spaces and $T\colon U \to V$ and $S\colon V \to W$ are linear. Prove that $\|S \circ T\| \leq \|S\|\,\|T\|$.

28 Suppose V and W are normed vector spaces and $T\colon V \to W$ is a linear map. Prove that the following are equivalent:

- T is bounded.
- There exists $f \in V$ such that T is continuous at f.
- T is uniformly continuous (which means that for every $\varepsilon > 0$, there exists $\delta > 0$ such that $\|Tf - Tg\| < \varepsilon$ for all $f, g \in V$ with $\|f - g\| < \delta$).
- $T^{-1}\big(B(0, r)\big)$ is an open subset of V for some $r > 0$.

29 Suppose V is a normed vector space and φ is a linear functional on V. Suppose $\alpha \in \mathbf{F} \setminus \{0\}$. Prove that the following are equivalent:

- φ is a bounded linear functional.
- $\varphi^{-1}(\alpha)$ is a closed subset of V.
- $\overline{\varphi^{-1}(\alpha)} \neq V$.

30 Suppose φ is a linear functional on a vector space V. Prove that if U is a subspace of V such that $\operatorname{null} \varphi \subset U$, then $U = \operatorname{null} \varphi$ or $U = V$.

31 Suppose φ and ψ are linear functionals on the same vector space. Prove that

$$\operatorname{null} \varphi \subset \operatorname{null} \psi$$

if and only if there exists $\alpha \in \mathbf{F}$ such that $\psi = \alpha\varphi$.

For the next three exercises, $\mathbf{F}^n$ should be endowed with the norm $\|\cdot\|_\infty$ as defined in Example 6.16.

32 Suppose V is a normed vector space, $n \in \mathbf{Z}^+$, and $T\colon V \to \mathbf{F}^n$ is linear. Prove that T is a bounded linear map if and only if $\operatorname{null} T$ is a closed subspace of V.

33 Suppose $n \in \mathbf{Z}^+$ and V is a normed vector space. Prove that every linear map from $\mathbf{F}^n$ to V is continuous.

34 Suppose $n \in \mathbf{Z}^+$, V is a normed vector space, and $T\colon \mathbf{F}^n \to V$ is a linear map that is one-to-one and onto V.

(a) Use the Bolzano–Weierstrass Theorem (see 0.73 in the Appendix) to show that
$$\inf\{\|Tx\| : x \in \mathbf{F}^n \text{ and } \|x\|_\infty = 1\} > 0.$$

(b) Prove that $T^{-1}\colon V \to \mathbf{F}^n$ is a bounded linear map.

35 Suppose $n \in \mathbf{Z}^+$.

(a) Prove that all norms on $\mathbf{F}^n$ have the same convergent sequences, the same open sets, and the same closed sets.

(b) Prove that all norms on $\mathbf{F}^n$ make $\mathbf{F}^n$ into a Banach space.