Chapter
5

Product Measures

Lebesgue measure on $\mathbf{R}$ generalizes the notion of the length of an interval. In this chapter, we will see how two-dimensional Lebesgue measure on $\mathbf{R}^2$ generalizes the notion of the area of a rectangle. More generally, we will construct new measures that are the products of two measures.

Once these new measures have been constructed, the question arises of how to compute integrals with respect to these new measures. Beautiful theorems proved in the first decade of the twentieth century will allow us to compute integrals with respect to product measures as iterated integrals involving the two measures that produced the product.

Main building of Scuola Normale Superiore di Pisa, the university in Pisa, Italy, where Guido Fubini (1879–1943) received his PhD in 1900. In 1907 Fubini proved that under reasonable conditions, an integral with respect to a product measure can be computed as an iterated integral and that the order of integration can be switched. Leonida Tonelli (1885–1943) also taught for many years in Pisa; he also discovered and proved an important theorem about interchanging the order of integration in an iterated integral.

5A Products of Measure Spaces

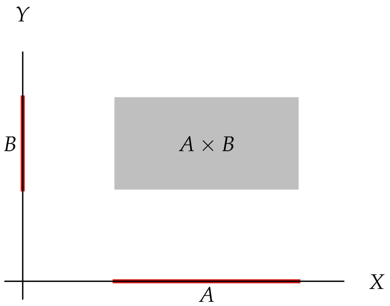
Products of σ-Algebras

Our first step in constructing product measures is to construct the product of two σ-algebras. We begin with the following definition.

5.1 Definition ***rectangle***

Suppose X and Y are sets. A *rectangle* in $X \times Y$ is a set of the form $A \times B$, where $A \subset X$ and $B \subset Y$.

Keep the figure shown here in mind when thinking of a rectangle in the sense defined above. However, remember that A and B need not be intervals as shown in the figure. Indeed, the concept of an interval makes no sense in the generality of arbitrary sets.



Now we can define the product of two σ-algebras.

5.2 Definition ***product of two σ-algebras; $\mathcal{S} \otimes \mathcal{T}$***

Suppose $(X, \mathcal{S})$ and $(Y, \mathcal{T})$ are measurable spaces. Then

- the *product* $\mathcal{S} \otimes \mathcal{T}$ is defined to be the smallest σ-algebra on $X \times Y$ that contains
$$\{A \times B : A \in \mathcal{S}, B \in \mathcal{T}\};$$
- a *measurable rectangle* in $\mathcal{S} \otimes \mathcal{T}$ is a set of the form $A \times B$, where $A \in \mathcal{S}$ and $B \in \mathcal{T}$.

Using the terminology introduced in the second bullet point above, we can say that $\mathcal{S} \otimes \mathcal{T}$ is the smallest σ-algebra containing all the measurable rectangles in $\mathcal{S} \otimes \mathcal{T}$. Exercise 1 in this section asks you to show that the measurable rectangles in $\mathcal{S} \otimes \mathcal{T}$ are the only rectangles in $X \times Y$ that are in $\mathcal{S} \otimes \mathcal{T}$.

The notation $\mathcal{S} \times \mathcal{T}$ is not used because $\mathcal{S}$ and $\mathcal{T}$ are sets (of sets), and thus the notation $\mathcal{S} \times \mathcal{T}$ already is defined to mean the set of all ordered pairs of the form (A, B), where $A \in \mathcal{S}$ and $B \in \mathcal{T}$.

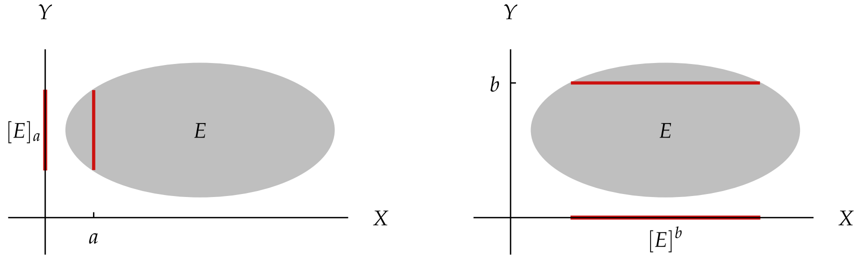
The notion of cross sections will play a crucial role in our development of product measures. First we define cross sections of sets, and then we will define cross sections of functions.

5.3 Definition ***cross sections of sets; $[E]_a$ and $[E]^b$***

Suppose X and Y are sets and $E \subset X \times Y$. Then for $a \in X$ and $b \in Y$, the *cross sections* $[E]_a$ and $[E]^b$ are defined by

$$[E]_a = \{y \in Y : (a, y) \in E\} \quad \text{and} \quad [E]^b = \{x \in X : (x, b) \in E\}.$$

5.4 Example *cross sections of a subset of $X \times Y$*



5.5 Example *cross sections of rectangles*

Suppose X and Y are sets and $A \subset X$ and $B \subset Y$. If $a \in X$ and $b \in Y$, then

$$[A \times B]_a = \begin{cases} B & \text{if } a \in A, \\ \varnothing & \text{if } a \notin A \end{cases} \quad \text{and} \quad [A \times B]^b = \begin{cases} A & \text{if } b \in B, \\ \varnothing & \text{if } b \notin B, \end{cases}$$

as you should verify.

The next result shows that cross sections preserve measurability.

5.6 ***Cross sections of measurable sets are measurable***

Suppose $\mathcal{S}$ is a σ-algebra on X and $\mathcal{T}$ is a σ-algebra on Y. If $E \in \mathcal{S} \otimes \mathcal{T}$, then

$$[E]_a \in \mathcal{T} \text{ for every } a \in X \quad \text{and} \quad [E]^b \in \mathcal{S} \text{ for every } b \in Y.$$

Proof Let $\mathcal{E}$ denote the collection of subsets E of $X \times Y$ for which the conclusion of this result holds. Then $A \times B \in \mathcal{E}$ for all $A \in \mathcal{S}$ and all $B \in \mathcal{T}$ (by Example 5.5).

The collection $\mathcal{E}$ is closed under complementation and countable unions because

$$[(X \times Y) \setminus E]_a = Y \setminus [E]_a$$

and

$$[E_1 \cup E_2 \cup \cdots]_a = [E_1]_a \cup [E_2]_a \cup \cdots$$

for all subsets $E, E_1, E_2, \ldots$ of $X \times Y$ and all $a \in X$, as you should verify, with similar statements holding for cross sections with respect to all $b \in Y$.

Because $\mathcal{E}$ is a σ-algebra containing all the measurable rectangles in $\mathcal{S} \otimes \mathcal{T}$, we conclude that $\mathcal{E}$ contains $\mathcal{S} \otimes \mathcal{T}$. ∎

Now we define cross sections of functions.

5.7 Definition ***cross sections of functions; $[f]_a$ and $[f]^b$***

Suppose X and Y are sets and $f\colon X \times Y \to \mathbf{R}$ is a function. Then for $a \in X$ and $b \in Y$, the cross section functions $[f]_a\colon Y \to \mathbf{R}$ and $[f]^b\colon X \to \mathbf{R}$ are defined by

$$[f]_a(y) = f(a, y) \text{ for } y \in Y \quad \text{and} \quad [f]^b(x) = f(x, b) \text{ for } x \in X.$$

5.8 Example *cross sections*

- Suppose $f\colon \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is defined by $f(x, y) = 5x^2 + y^3$. Then

  $$[f]_2(y) = 20 + y^3 \quad \text{and} \quad [f]^3(x) = 5x^2 + 27$$

  for all $y \in \mathbf{R}$ and all $x \in \mathbf{R}$, as you should verify.

- Suppose X and Y are sets and $A \subset X$ and $B \subset Y$. If $a \in X$ and $b \in Y$, then

  $$[\chi_{A \times B}]_a = \chi_A(a)\chi_B \quad \text{and} \quad [\chi_{A \times B}]^b = \chi_B(b)\chi_A,$$

  as you should verify.

The next result shows that cross sections preserve measurability, this time in the context of functions rather than sets.

5.9 ***Cross sections of measurable functions are measurable***

Suppose $\mathcal{S}$ is a σ-algebra on X and $\mathcal{T}$ is a σ-algebra on Y. Suppose $f\colon X \times Y \to \mathbf{R}$ is an $\mathcal{S} \otimes \mathcal{T}$-measurable function. Then

$$[f]_a \text{ is a } \mathcal{T}\text{-measurable function on } Y \text{ for every } a \in X$$

and

$$[f]^b \text{ is an } \mathcal{S}\text{-measurable function on } X \text{ for every } b \in Y.$$

Proof Suppose D is a Borel subset of $\mathbf{R}$ and $a \in X$. If $y \in Y$, then

$$\begin{aligned} y \in ([f]_a)^{-1}(D) &\iff [f]_a(y) \in D \\ &\iff f(a, y) \in D \\ &\iff (a, y) \in f^{-1}(D) \\ &\iff y \in [f^{-1}(D)]_a. \end{aligned}$$

Thus

$$([f]_a)^{-1}(D) = [f^{-1}(D)]_a.$$

Because f is an $\mathcal{S} \otimes \mathcal{T}$-measurable function, $f^{-1}(D) \in \mathcal{S} \otimes \mathcal{T}$. Thus the equation above and 5.6 imply that $([f]_a)^{-1}(D) \in \mathcal{T}$. Hence $[f]_a$ is a $\mathcal{T}$-measurable function.

The same ideas show that $[f]^b$ is an $\mathcal{S}$-measurable function for every $b \in Y$. ■

Monotone Class Theorem

The following standard two-step technique often works to prove that every set in a σ-algebra has a certain property:

1. show that every set in a collection of sets that generates the σ-algebra has the property;
2. show that the collection of sets that has the property is a σ-algebra.

For example, the proof of 5.6 used the technique above—first we showed that every measurable rectangle in $\mathcal{S} \otimes \mathcal{T}$ has the desired property, then we showed that the collection of sets that has the desired property is a σ-algebra (this completed the proof because $\mathcal{S} \otimes \mathcal{T}$ is the smallest σ-algebra containing the measurable rectangles).

The technique outlined above should be used when possible. However, in some situations there seems to be no reasonable way to verify that the collection of sets with the desired property is a σ-algebra. We will encounter this situation in the next subsection. To deal with it, we need to introduce another technique that involves what are called monotone classes.

The following definition will be used in our main theorem about monotone classes.

5.10 Definition *algebra*

Suppose W is a set and $\mathcal{A}$ is a set of subsets of W. Then $\mathcal{A}$ is called an *algebra* on W if the following three conditions are satisfied:

- $\varnothing \in \mathcal{A}$;
- if $E \in \mathcal{A}$, then $W \setminus E \in \mathcal{A}$;
- if E and F are elements of $\mathcal{A}$, then $E \cup F \in \mathcal{A}$.

Thus an algebra is closed under complementation and under finite unions; a σ-algebra is closed under complementation and countable unions.

5.11 Example *finite unions of intervals is an algebra*

Suppose $\mathcal{A}$ is the collection of all finite unions of intervals of $\mathbf{R}$. Here we are including all intervals—open intervals, closed intervals, sets consisting of only a single point, and intervals that are neither open nor closed (see Definition 0.42 in the Appendix for the definition of interval).

Clearly $\mathcal{A}$ is closed under finite unions. You should also verify that $\mathcal{A}$ is closed under complementation. Thus $\mathcal{A}$ is an algebra on $\mathbf{R}$.

5.12 Example *countable unions of intervals is not an algebra*

Suppose $\mathcal{A}$ is the collection of all countable unions of intervals of $\mathbf{R}$.

Clearly $\mathcal{A}$ is closed under finite unions (and also under countable unions). You should verify that $\mathcal{A}$ is not closed under complementation. Thus $\mathcal{A}$ is neither an algebra nor a σ-algebra on $\mathbf{R}$.

The following result provides an example of an algebra that we will exploit.

5.13 *The set of finite unions of measurable rectangles is an algebra*

Suppose $(X, \mathcal{S})$ and $(Y, \mathcal{T})$ are measurable spaces. Then

(a) the set of finite unions of measurable rectangles in $\mathcal{S} \otimes \mathcal{T}$ is an algebra on $X \times Y$;

(b) every finite union of measurable rectangles in $\mathcal{S} \otimes \mathcal{T}$ can be written as a finite union of disjoint measurable rectangles in $\mathcal{S} \otimes \mathcal{T}$.

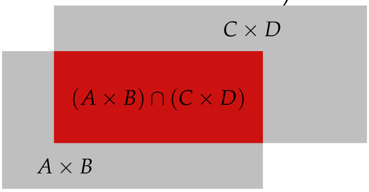
Proof Let $\mathcal{A}$ denote the set of finite unions of measurable rectangles in $\mathcal{S} \otimes \mathcal{T}$. Obviously $\mathcal{A}$ is closed under finite unions.

The collection $\mathcal{A}$ is also closed under finite intersections. To verify this claim, note that if $A_1, \dots, A_n, C_1, \dots, C_m \in \mathcal{S}$ and $B_1, \dots, B_n, D_1, \dots, D_m \in \mathcal{T}$, then

$$\Big((A_1 \times B_1) \cup \dots \cup (A_n \times B_n)\Big) \bigcap \Big((C_1 \times D_1) \cup \dots \cup (C_m \times D_m)\Big)$$

$$= \bigcup_{j=1}^{n} \bigcup_{k=1}^{m} \Big((A_j \times B_j) \cap (C_k \times D_k)\Big)$$

$$= \bigcup_{j=1}^{n} \bigcup_{k=1}^{m} \Big((A_j \cap C_k) \times (B_j \cap D_k)\Big),$$



Intersection of two rectangles is a rectangle.

which implies that $\mathcal{A}$ is closed under finite intersections.

If $A \in \mathcal{S}$ and $B \in \mathcal{T}$, then

$$(X \times Y) \setminus (A \times B) = \Big((X \setminus A) \times Y\Big) \cup \Big(X \times (Y \setminus B)\Big).$$

Hence the complement of each measurable rectangle in $\mathcal{S} \otimes \mathcal{T}$ is in $\mathcal{A}$. Thus the complement of a finite union of measurable rectangles in $\mathcal{S} \otimes \mathcal{T}$ is in $\mathcal{A}$ (use De Morgan's Laws and the result in the previous paragraph that $\mathcal{A}$ is closed under finite intersections). In other words, $\mathcal{A}$ is closed under complementation, completing the proof of (a).

To prove (b), note that if $A \times B$ and $C \times D$ are measurable rectangles in $\mathcal{S} \otimes \mathcal{T}$, then (as can be verified in the figure above)

5.14 $$(A \times B) \cup (C \times D) = (A \times B) \cup \Big(C \times (D \setminus B)\Big) \cup \Big((C \setminus A) \times (B \cap D)\Big).$$

The equation above writes the union of two measurable rectangles in $\mathcal{S} \otimes \mathcal{T}$ as the union of three disjoint measurable rectangles in $\mathcal{S} \otimes \mathcal{T}$.

Now consider any finite union of measurable rectangles in $\mathcal{S} \otimes \mathcal{T}$. If this is not a disjoint union, then choose any nondisjoint pair of measurable rectangles in the union and replace those two measurable rectangles with the union of three disjoint measurable rectangles as in 5.14. Iterate this process until obtaining a disjoint union of measurable rectangles. ∎

Now we define a monotone class as a collection of sets that is closed under countable increasing unions and under countable decreasing intersections.

5.15 Definition *monotone class*

Suppose W is a set and $\mathcal{M}$ is a set of subsets of W. Then $\mathcal{M}$ is called a *monotone class* on W if the following two conditions are satisfied:

- If $E_1 \subset E_2 \subset \cdots$ is an increasing sequence of sets in $\mathcal{M}$, then $\bigcup_{k=1}^{\infty} E_k \in \mathcal{M}$;
- If $E_1 \supset E_2 \supset \cdots$ is a decreasing sequence of sets in $\mathcal{M}$, then $\bigcap_{k=1}^{\infty} E_k \in \mathcal{M}$.

Clearly every σ-algebra is a monotone class. However, some monotone classes are not closed under even finite unions, as shown by the next example.

5.16 Example *a monotone class that is not an algebra*

Suppose $\mathcal{A}$ is the collection of all intervals of $\mathbf{R}$. Then $\mathcal{A}$ is closed under countable increasing unions and countable decreasing intersections. Thus $\mathcal{A}$ is a monotone class on $\mathbf{R}$. However, $\mathcal{A}$ is not closed under finite unions, and $\mathcal{A}$ is not closed under complementation. Thus $\mathcal{A}$ is neither an algebra nor a σ-algebra on $\mathbf{R}$.

If $\mathcal{A}$ is a collection of subsets of some set W, then the intersection of all monotone classes on W that contain A is a monotone class that contains $\mathcal{A}$. Thus this intersection is the smallest monotone class on W that contains $\mathcal{A}$.

The next result provides a useful tool when the standard technique to show that every set in a σ-algebra has a certain property does not work.

5.17 ***Monotone Class Theorem***

Suppose $\mathcal{A}$ is an algebra on a set W. Then the smallest σ-algebra containing $\mathcal{A}$ is the smallest monotone class containing $\mathcal{A}$.

Proof Let $\mathcal{M}$ denote the smallest monotone class containing $\mathcal{A}$. Because every σ-algebra is a monotone class, $\mathcal{M}$ is contained in the smallest σ-algebra containing $\mathcal{A}$.

To prove the inclusion in the other direction, first suppose $A \in \mathcal{A}$. Let

$$\mathcal{E} = \{E \in \mathcal{M} : A \cup E \in \mathcal{M}\}.$$

Then $\mathcal{A} \subset \mathcal{E}$ (because the union of two sets in $\mathcal{A}$ is in $\mathcal{A}$). A moment's thought shows that $\mathcal{E}$ is a monotone class. Thus the smallest monotone class that contains $\mathcal{A}$ is contained in $\mathcal{E}$, meaning that $\mathcal{M} \subset \mathcal{E}$. Hence we have proved that $A \cup E \in \mathcal{M}$ for every $E \in \mathcal{M}$.

Now let

$$\mathcal{D} = \{D \in \mathcal{M} : D \cup E \in \mathcal{M} \text{ for all } E \in \mathcal{M}\}.$$

The previous paragraph shows that $\mathcal{A} \subset \mathcal{D}$. A moment's thought again shows that $\mathcal{D}$ is a monotone class. Thus, as in the previous paragraph, we conclude that $\mathcal{M} \subset \mathcal{D}$. Hence we have proved that $D \cup E \in \mathcal{M}$ for all $D, E \in \mathcal{M}$.

The paragraph above shows that the monotone class $\mathcal{M}$ is closed under finite unions. Now if $E_1, E_2, \ldots \in \mathcal{M}$, then

$$E_1 \cup E_2 \cup E_3 \cup \cdots = E_1 \cup (E_1 \cup E_2) \cup (E_1 \cup E_2 \cup E_3) \cup \cdots ,$$

which is an increasing union of a sequence of sets in $\mathcal{M}$ (by the previous paragraph). We conclude that $\mathcal{M}$ is closed under countable unions.

Finally, let

$$\mathcal{M}' = \{E \in \mathcal{M} : X \setminus E \in \mathcal{M}\}.$$

Then $\mathcal{A} \subset \mathcal{M}'$ (because $\mathcal{A}$ is closed under complementation). Once again, you should verify that $\mathcal{M}'$ is a monotone class. Thus $\mathcal{M} \subset M'$. We conclude that $\mathcal{M}$ is closed under complementation.

The two previous paragraphs show that $\mathcal{M}$ is closed under countable unions and under complementation. Thus $\mathcal{M}$ is a σ-algebra that contains $\mathcal{A}$. Hence $\mathcal{M}$ contains the smallest σ-algebra containing $\mathcal{A}$, completing the proof. ■

Products of Measures

The following definitions will be useful.

5.18 Definition ***finite measure; σ-finite measure***

- A measure μ on a measurable space $(X, \mathcal{S})$ is called *finite* if $\mu(X) < \infty$.
- A measure is called *σ-finite* if the whole space can be written as the countable union of sets with finite measure
- More precisely, a measure μ on a measurable space $(X, \mathcal{S})$ is called *σ-finite* if there exists a sequence $X_1, X_2, \ldots$ of sets in $\mathcal{S}$ such that

$$X = \bigcup_{k=1}^{\infty} X_k \quad \text{and} \quad \mu(X_k) < \infty \text{ for every } k \in \mathbf{Z}^+.$$

5.19 Example *finite and σ-finite measures*

- Lebesgue measure on the interval $[0, 1]$ is a finite measure.
- Lebesgue measure on $\mathbf{R}$ is not a finite measure but is a σ-finite measure.
- Counting measure on $\mathbf{R}$ is not a σ-finite measure (because the countable union of finite sets is a countable set).

The next result will allow us to define the product of two σ-finite measures.

5.20 *Measure of cross section is a measurable function*

Suppose $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \nu)$ are σ-finite measure spaces. If $E \in \mathcal{S} \otimes \mathcal{T}$, then

(a) $x \mapsto \nu([E]_x)$ is an $\mathcal{S}$-measurable function on X;

(b) $y \mapsto \mu([E]^y)$ is a $\mathcal{T}$-measurable function on Y.

Proof We will prove (a). If $E \in \mathcal{S} \otimes \mathcal{T}$, then $[E]_x \in \mathcal{T}$ for every $x \in X$ (by 5.6); thus the function $x \mapsto \nu([E]_x)$ is well defined on X.

We first consider the case where ν is a finite measure. Let

$$\mathcal{M} = \{E \in \mathcal{S} \otimes \mathcal{T} : x \mapsto \nu([E]_x) \text{ is an } \mathcal{S}\text{-measurable function on } X\}.$$

We need to prove that $\mathcal{M} = \mathcal{S} \otimes \mathcal{T}$.

If $A \in \mathcal{S}$ and $B \in \mathcal{T}$, then $\nu([A \times B]_x) = \nu(B)\chi_A(x)$ for every $x \in X$ (by Example 5.5). Thus the function $x \mapsto \nu([A \times B]_x)$ equals the function $\nu(B)\chi_A$ (as a function on X), which is an $\mathcal{S}$-measurable function on X. Hence $\mathcal{M}$ contains all the measurable rectangles in $\mathcal{S} \otimes \mathcal{T}$.

Let $\mathcal{A}$ denote the set of finite unions of measurable rectangles in $\mathcal{S} \otimes \mathcal{T}$. Suppose $E \in \mathcal{A}$. Then by 5.13(b), E is a union of disjoint measurable rectangles $E_1, \dots, E_n$. Thus

$$\begin{aligned}
\nu([E]_x) &= \nu([E_1 \cup \dots \cup E_n]_x) \\
&= \nu([E_1]_x \cup \dots \cup [E_n]_x) \\
&= \nu([E_1]_x) + \dots + \nu([E_n]_x),
\end{aligned}$$

where the last equality holds because ν is a measure and $[E_1]_x, \dots, [E_n]_x$ are disjoint. The equation above, when combined with the conclusion of the previous paragraph, shows that $x \mapsto \nu([E]_x)$ is a finite sum of $\mathcal{S}$-measurable functions and thus is an $\mathcal{S}$-measurable function. Hence $E \in \mathcal{M}$. We have now shown that $\mathcal{A} \subset \mathcal{M}$.

Our next goal is to show that $\mathcal{M}$ is a monotone class on $X \times Y$. To do this, first suppose that $E_1 \subset E_2 \subset \cdots$ is an increasing sequence of sets in $\mathcal{M}$. Then

$$\begin{aligned}
\nu\Big(\Big[\bigcup_{k=1}^{\infty} E_k\Big]_x\Big) &= \nu\Big(\bigcup_{k=1}^{\infty} ([E_k]_x)\Big) \\
&= \lim_{k \to \infty} \nu([E_k]_x),
\end{aligned}$$

where we have used 2.58. Because the pointwise limit of $\mathcal{S}$-measurable functions is $\mathcal{S}$-measurable (by 2.47), the equation above shows that $x \mapsto \nu\big([\bigcup_{k=1}^{\infty} E_k]_x\big)$ is an $\mathcal{S}$-measurable function. Hence $\bigcup_{k=1}^{\infty} E_k \in \mathcal{M}$. We have now shown that $\mathcal{M}$ is closed under countable increasing unions.

Now suppose that $E_1 \supset E_2 \supset \cdots$ is a decreasing sequence of sets in $\mathcal{M}$. Then

$$\begin{aligned}\nu\Big([\bigcap_{k=1}^{\infty} E_k]_x\Big) &= \nu\Big(\bigcap_{k=1}^{\infty}([E_k]_x)\Big)\\ &= \lim_{k\to\infty} \nu([E_k]_x),\end{aligned}$$

where we have used 2.59 (this is where we use the assumption that ν is a finite measure). Because the pointwise limit of $\mathcal{S}$-measurable functions is $\mathcal{S}$-measurable (by 2.47), the equation above shows that $x \mapsto \nu\big([\bigcap_{k=1}^{\infty} E_k]_x\big)$ is an $\mathcal{S}$-measurable function. Hence $\bigcap_{k=1}^{\infty} E_k \in \mathcal{M}$. We have now shown that $\mathcal{M}$ is closed under countable decreasing intersections.

We have shown that $\mathcal{M}$ is a monotone class that contains the algebra $\mathcal{A}$ of all finite unions of measurable rectangles in $\mathcal{S} \otimes \mathcal{T}$ [by 5.13(a), $\mathcal{A}$ is indeed an algebra]. The Monotone Class Theorem (5.17) implies that $\mathcal{M}$ contains the smallest σ-algebra containing $\mathcal{A}$. In other words, $\mathcal{M}$ contains $\mathcal{S} \otimes \mathcal{T}$. This conclusion completes the proof of (a) in the case where ν is a finite measure.

Now consider the case where ν is a σ-finite measure. Thus there exists a sequence $Y_1, Y_2, \ldots$ of sets in $\mathcal{T}$ such that $\bigcup_{k=1}^{\infty} Y_k = Y$ and $\nu(Y_k) < \infty$ for each $k \in \mathbf{Z}^+$. Replacing each Y_k by $Y_1 \cup \cdots \cup Y_k$, we can assume that $Y_1 \subset Y_2 \subset \cdots$. If $E \in \mathcal{S} \otimes \mathcal{T}$, then

$$\nu([E]_x) = \lim_{k\to\infty} \nu([E \cap (X \times Y_k)]_x).$$

The function $x \mapsto \nu([E \cap (X \times Y_k)]_x)$ is an $\mathcal{S}$-measurable function on X, as follows by considering the finite measure obtained by restricting ν to the σ-algebra on Y_k consisting of sets in $\mathcal{T}$ that are contained in Y_k. The equation above now implies that $x \mapsto \nu([E]_x)$ is an $\mathcal{S}$-measurable function on X, completing the proof of (a).

The proof of (b) is similar. ■

5.21 Definition *integration notation*

Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $g\colon X \to [-\infty, \infty]$ is a function. The notation

$$\int g(x)\, d\mu(x) \quad \text{means} \quad \int g\, d\mu,$$

where $d\mu(x)$ indicates that variables other than x should be treated as constants.

5.22 Example *integrals*

If λ is Lebesgue measure on $[0,4]$, then

$$\int_{[0,4]} (x^2 + y)\, d\lambda(y) = 4x^2 + 8 \quad \text{and} \quad \int_{[0,4]} (x^2 + y)\, d\lambda(x) = \frac{64}{3} + 4y.$$

The intent in the next definition is that $\int_X \int_Y f(x,y)\, d\nu(y)\, d\mu(x)$ is defined only when the inner integral and then the outer integral both make sense.

5.23 Definition ***iterated integrals***

Suppose $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \nu)$ are measure spaces and $f\colon X \times Y \to \mathbf{R}$ is a function. Then

$$\int_X \int_Y f(x,y)\, d\nu(y)\, d\mu(x) \quad \text{means} \quad \int_X \Big(\int_Y f(x,y)\, d\nu(y) \Big)\, d\mu(x).$$

In other words, to compute $\int_X \int_Y f(x,y)\, d\nu(y)\, d\mu(x)$, first (temporarily) fix $x \in X$ and compute $\int_Y f(x,y)\, d\nu(y)$ [if this integral makes sense]. Then compute the integral with respect to μ of the function $x \mapsto \int_Y f(x,y)\, d\nu(y)$ [if this integral makes sense].

5.24 Example *iterated integrals*

If λ is Lebesgue measure on $[0,4]$, then

$$\begin{aligned}\int_{[0,4]} \int_{[0,4]} (x^2 + y)\, d\lambda(y)\, d\lambda(x) &= \int_{[0,4]} (4x^2 + 8)\, d\lambda(x) \\ &= \frac{352}{3}\end{aligned}$$

and

$$\begin{aligned}\int_{[0,4]} \int_{[0,4]} (x^2 + y)\, d\lambda(x)\, d\lambda(y) &= \int_{[0,4]} \Big(\frac{64}{3} + 4y\Big)\, d\lambda(y) \\ &= \frac{352}{3}.\end{aligned}$$

The two iterated integrals in this example turned out to both equal $\frac{352}{3}$, even though they do not look alike in the intermediate step of the evaluation. As we will see in the next section, this equality of integrals when changing the order of integration is not a coincidence.

The definition of $(\mu \times \nu)(E)$ given below makes sense because the inner integral below equals $\nu([E]_x)$, which makes sense by 5.6 (or use 5.9), and then the outer integral makes sense by 5.20(a).

The restriction in the definition below to σ-finite measures is not bothersome because the main results we seek are not valid without this hypothesis (see Example 5.30 in the next section).

5.25 Definition ***product of two measures; $\mu \times \nu$***

Suppose $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \nu)$ are σ-finite measure spaces. For $E \in \mathcal{S} \otimes \mathcal{T}$, define $(\mu \times \nu)(E)$ by

$$(\mu \times \nu)(E) = \int_X \int_Y \chi_E(x,y)\, d\nu(y)\, d\mu(x).$$

5.26 Example *measure of a rectangle*

Suppose $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \nu)$ are σ-finite measure spaces. If $A \in \mathcal{S}$ and $B \in \mathcal{T}$, then

$$\begin{aligned}(\mu \times \nu)(A \times B) &= \int_X \int_Y \chi_{A \times B}(x, y)\, d\nu(y)\, d\mu(x) \\ &= \int_X \nu(B)\chi_A(x)\, d\mu(x) \\ &= \mu(A)\nu(B).\end{aligned}$$

In other words, the product measure of a rectangle is the product of the measures of the corresponding sets.

For $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \nu)$ σ-finite measure spaces, we defined the product $\mu \times \nu$ to be a function from $\mathcal{S} \otimes \mathcal{T}$ to $[0, \infty]$; see 5.25. Now we show that this function is a measure.

5.27 *Product of two measures is a measure*

Suppose $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \nu)$ are σ-finite measure spaces. Then $\mu \times \nu$ is a measure on $(X \times Y, \mathcal{S} \otimes \mathcal{T})$.

Proof Clearly $(\mu \times \nu)(\varnothing) = 0$.

Suppose $E_1, E_2, \ldots$ is a disjoint sequence of sets in $\mathcal{S} \otimes \mathcal{T}$. Then

$$\begin{aligned}(\mu \times \nu)\Big(\bigcup_{k=1}^{\infty} E_k\Big) &= \int_X \nu\Big(\big[\bigcup_{k=1}^{\infty} E_k\big]_x\Big)\, d\mu(x) \\ &= \int_X \nu\Big(\bigcup_{k=1}^{\infty} ([E_k]_x)\Big)\, d\mu(x) \\ &= \int_X \Big(\sum_{k=1}^{\infty} \nu([E_k]_x)\Big)\, d\mu(x) \\ &= \sum_{k=1}^{\infty} \int_X \nu([E_k]_x)\, d\mu(x) \\ &= \sum_{k=1}^{\infty} (\mu \times \nu)(E_k),\end{aligned}$$

where the fourth equality follows from the Monotone Convergence Theorem (3.11; or see Exercise 10 in Section 3A). The equation above shows that $\mu \times \nu$ satisfies the countable additivity condition required for a measure. ∎

EXERCISES 5A

1 Suppose $(X, \mathcal{S})$ and $(Y, \mathcal{T})$ are measurable spaces. Prove that if A is a nonempty subset of X and B is a nonempty subset of Y such that $A \times B \in \mathcal{S} \otimes \mathcal{T}$, then $A \in \mathcal{S}$ and $B \in \mathcal{T}$.

2 Suppose $\mathcal{S}$ is the σ-algebra on $\mathbf{R}$ consisting of all subsets of $\mathbf{R}$ that are either countable or have a countable complement in $\mathbf{R}$. Is

$$\{(x, x) : x \in \mathbf{R}\}$$

in the σ-algebra $\mathcal{S} \otimes \mathcal{S}$?

3 Suppose $(X, \mathcal{S})$ is a measurable space. Prove that if $E \in \mathcal{S} \otimes \mathcal{S}$, then

$$\{x \in X : (x, x) \in E\} \in \mathcal{S}.$$

4 Let $\mathcal{B}$ denote the σ-algebra of Borel subsets of $\mathbf{R}$. Show that there exists a set $E \subset \mathbf{R} \times \mathbf{R}$ such that $[E]_a \in \mathcal{B}$ and $[E]^a \in \mathcal{B}$ for every $a \in \mathbf{R}$, but $E \notin \mathcal{B} \otimes \mathcal{B}$.

5 Verify the assertion in Example 5.11 that the collection of finite unions of intervals of $\mathbf{R}$ is closed under complementation.

6 Verify the assertion in Example 5.12 that the collection of countable unions of intervals of $\mathbf{R}$ is not closed under complementation.

7 Suppose $\mathcal{A}$ is a nonempty collection of subsets of a set W. Show that $\mathcal{A}$ is an algebra on W if and only if $\mathcal{A}$ is closed under finite intersections and under complementation.

8 Suppose μ is a measure on a measurable space $(X, \mathcal{S})$. Prove that the following are equivalent:

(a) The measure μ is σ-finite.

(b) There exists an increasing sequence $X_1 \subset X_2 \subset \cdots$ of sets in $\mathcal{S}$ such that $X = \bigcup_{k=1}^{\infty} X_k$ and $\mu(X_k) < \infty$ for every $k \in \mathbf{Z}^+$.

(c) There exists a disjoint sequence $X_1, X_2, X_3, \ldots$ of sets in $\mathcal{S}$ such that $X = \bigcup_{k=1}^{\infty} X_k$ and $\mu(X_k) < \infty$ for every $k \in \mathbf{Z}^+$.

9 Suppose μ and ν are σ-finite measures. Prove that $\mu \times \nu$ is a σ-finite measure.

10 Suppose $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \nu)$ are σ-finite measure spaces. Prove that if ω is a measure on $\mathcal{S} \otimes \mathcal{T}$ such that $\omega(A \times B) = \mu(A)\nu(B)$ for all $A \in \mathcal{S}$ and all $B \in \mathcal{T}$, then $\omega = \mu \times \nu$.
[*The exercise above means that $\mu \times \nu$ is the unique measure on $\mathcal{S} \otimes \mathcal{T}$ that behaves as we expect on measurable rectangles.*]

5B Iterated Integrals

Tonelli's Theorem

Relook at Example 5.24 in the previous section and notice that the value of the iterated integral was unchanged when we switched the order of integration, even though switching the order of integration led to different intermediate results. Our next result states that the order of integration can be switched if the function being integrated is nonnegative and the measures are σ-finite.

5.28 *Tonelli's Theorem*

Suppose $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \nu)$ are σ-finite measure spaces. Suppose $f \colon X \times Y \to [0, \infty]$ is $\mathcal{S} \otimes \mathcal{T}$-measurable. Then

(a) $x \mapsto \int_Y f(x, y)\, d\nu(y)$ is an $\mathcal{S}$-measurable function on X,

(b) $y \mapsto \int_X f(x, y)\, d\mu(x)$ is a $\mathcal{T}$-measurable function on Y,

and

$$\int_{X\times Y} f\, d(\mu \times \nu) = \int_X \int_Y f(x, y)\, d\nu(y)\, d\mu(x) = \int_Y \int_X f(x, y)\, d\mu(x)\, d\nu(y).$$

Proof We begin by considering the special case where $f = \chi_E$ for some $E \in \mathcal{S} \otimes \mathcal{T}$. In this case,

$$\int_Y \chi_E(x, y)\, d\nu(y) = \nu([E]_x) \text{ for every } x \in X$$

and

$$\int_X \chi_E(x, y)\, d\mu(x) = \mu([E]^y) \text{ for every } y \in Y.$$

Thus (a) and (b) hold in this case by 5.20.

First assume that μ and ν are finite measures. Let

$$\mathcal{M} = \Big\{E \in \mathcal{S} \otimes \mathcal{T} \colon \int_X \int_Y \chi_E(x, y)\, d\nu(y)\, d\mu(x) = \int_Y \int_X \chi_E(x, y)\, d\mu(x)\, d\nu(y)\Big\}.$$

If $A \in \mathcal{S}$ and $B \in \mathcal{T}$, then $A \times B \in \mathcal{M}$ because both sides of the equation defining $\mathcal{M}$ equal $\mu(A)\nu(B)$.

Let $\mathcal{A}$ denote the set of finite unions of measurable rectangles in $\mathcal{S} \otimes \mathcal{T}$. Then 5.13(b) implies that every element of $\mathcal{A}$ is a disjoint union of measurable rectangles in $\mathcal{S} \otimes \mathcal{T}$. The previous paragraph now implies $\mathcal{A} \subset \mathcal{M}$.

The Monotone Convergence Theorem (3.11) implies that $\mathcal{M}$ is closed under countable increasing unions. The Bounded Convergence Theorem (3.26) implies that $\mathcal{M}$ is closed under countable decreasing intersections (this is where we use the assumption that μ and ν are finite measures).

We have shown that $\mathcal{M}$ is a monotone class that contains the algebra $\mathcal{A}$ of all finite unions of measurable rectangles in $\mathcal{S} \otimes \mathcal{T}$ [by 5.13(a), $\mathcal{A}$ is indeed an algebra].

The Monotone Class Theorem (5.17) implies that $\mathcal{M}$ contains the smallest σ-algebra containing $\mathcal{A}$. In other words, $\mathcal{M}$ contains $\mathcal{S} \otimes \mathcal{T}$. Thus

5.29 $$\int_X \int_Y \chi_E(x,y)\, d\nu(y)\, d\mu(x) = \int_Y \int_X \chi_E(x,y)\, d\mu(x)\, d\nu(y)$$

for every $E \in \mathcal{S} \otimes \mathcal{T}$.

Now relax the assumption that μ and ν are finite measures. Write X as an increasing union of sets $X_1 \subset X_2 \subset \cdots$ in $\mathcal{S}$ with finite measure, and write Y as an increasing union of sets $Y_1 \subset Y_2 \subset \cdots$ in $\mathcal{T}$ with finite measure. Suppose $E \in \mathcal{S} \otimes \mathcal{T}$. Applying the finite-measure case to the situation where the measures and the σ-algebras are restricted to X_j and Y_k, we can conclude that 5.29 holds with E replaced by $E \cap (X_j \times Y_k)$ for all $j, k \in \mathbf{Z}^+$. Fix $k \in \mathbf{Z}^+$ and use the Monotone Convergence Theorem (3.11) to conclude that 5.29 holds with E replaced by $E \cap (X \times Y_k)$ for all $k \in \mathbf{Z}^+$. One more use of the Monotone Convergence Theorem then shows that

$$\int_{X \times Y} \chi_E\, d(\mu \times \nu) = \int_X \int_Y \chi_E(x,y)\, d\nu(y)\, d\mu(x) = \int_Y \int_X \chi_E(x,y)\, d\mu(x)\, d\nu(y)$$

for all $E \in \mathcal{S} \otimes \mathcal{T}$, where the first equality above comes from the definition of $(\mu \times \nu)(E)$ (see 5.25).

Now we turn from characteristic functions to the general case of an $\mathcal{S} \otimes \mathcal{T}$-measurable function $f \colon X \times Y \to [0, \infty]$. Define a sequence $f_1, f_2, \dots$ of simple $\mathcal{S} \otimes \mathcal{T}$-measurable functions from $X \times Y$ to $[0, \infty)$ by

$$f_k(x,y) = \begin{cases} \dfrac{m}{2^k} & \text{if } f(x,y) < k \text{ and } m \text{ is the integer with } f(x,y) \in \Big[\dfrac{m}{2^k}, \dfrac{m+1}{2^k}\Big), \\ k & \text{if } f(x,y) \geq k. \end{cases}$$

Note that

$$0 \leq f_1(x,y) \leq f_2(x,y) \leq f_3(x,y) \leq \cdots \quad \text{and} \quad \lim_{k \to \infty} f_k(x,y) = f(x,y)$$

for all $(x,y) \in X \times Y$.

Each f_k is a finite sum of functions of the form $c\chi_E$, where $c \in \mathbf{R}$ and $E \in \mathcal{S} \otimes \mathcal{T}$. Thus the conclusions of this theorem hold for each function f_k.

The Monotone Convergence Theorem implies that

$$\int_Y f(x,y)\, d\nu(y) = \lim_{k \to \infty} \int_Y f_k(x,y)\, d\nu(y)$$

for every $x \in X$. Thus the function $x \mapsto \int_Y f(x,y)\, d\nu(y)$ is the pointwise limit on X of a sequence of $\mathcal{S}$-measurable functions. Hence (a) holds, as does (b) for similar reasons.

The last line in the statement of this theorem holds for each f_k. The Monotone Convergence Theorem now implies that the last line in the statement of this theorem holds for f, completing the proof. ■

See Exercise 1 in this section for an example (with finite measures) showing that Tonelli's Theorem can fail without the hypothesis that the function being integrated is nonnegative. The next example shows that the hypothesis of σ-finite measures also cannot be eliminated.

5.30 **Example** *Tonelli's Theorem can fail without the hypothesis of σ-finite*

Suppose $\mathcal{B}$ is the σ-algebra of Borel subsets of $[0,1]$, λ is Lebesgue measure on $([0,1],\mathcal{B})$, and μ is counting measure on $([0,1],\mathcal{B})$. Let D denote the diagonal of $[0,1]\times[0,1]$; in other words,

$$D=\{(x,x):x\in[0,1]\}.$$

Then

$$\int_{[0,1]}\int_{[0,1]}\chi_D(x,y)\,d\mu(y)\,d\lambda(x)=\int_{[0,1]}1\,d\lambda=1,$$

but

$$\int_{[0,1]}\int_{[0,1]}\chi_D(x,y)\,d\lambda(x)\,d\mu(y)=\int_{[0,1]}0\,d\mu=0.$$

The following useful corollary of Tonelli's Theorem states that we can switch the order of summation in a double-sum of nonnegative numbers. Exercise 2 asks you to find a double-sum of real numbers in which switching the order of summation changes the value of the double sum.

5.31 ***Double sums of nonnegative numbers***

Suppose $\{x_{j,k}:j,k\in\mathbf{Z}^+\}$ is a doubly-indexed collection of nonnegative numbers. Then

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}x_{j,k}=\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}x_{j,k}.$$

Proof Apply Tonelli's Theorem (5.28) to $\mu\times\mu$, where μ is counting measure on $\mathbf{Z}^+$. ■

Fubini's Theorem

Our next goal is Fubini's Theorem, which has the same conclusions as Tonelli's Theorem but has a different hypothesis. Tonelli's Theorem requires that the function being integrated is nonnegative. Fubini's Theorem does not require nonnegativity but instead requires that the absolute value of the function being integrated has a finite integral. When using Fubini's Theorem, you will usually first use Tonelli's Theorem as applied to $|f|$ to verify the hypothesis of Fubini's Theorem.

Historically, Fubini's Theorem (proved in 1907) came before Tonelli's Theorem (proved in 1909). However, presenting Tonelli's Theorem first, as is done here, seems to lead to simpler proofs and better understanding. The hard work here went into proving Tonelli's Theorem; thus our proof of Fubini's Theorem consists mainly of bookkeeping details.

As you will see in the proof of Fubini's Theorem, the function in 5.32(a) is defined only for almost every $x \in X$ and the function in 5.32(b) is defined only for almost every $y \in Y$. For convenience, you can think of these functions as equaling 0 on the sets of measure 0 on which they are otherwise defined.

5.32 *Fubini's Theorem*

Suppose $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \nu)$ are σ-finite measure spaces. Suppose $f\colon X \times Y \to [-\infty, \infty]$ is $\mathcal{S} \otimes \mathcal{T}$-measurable and $\int_{X \times Y} |f| \, d(\mu \times \nu) < \infty$. Then

$$\int_Y |f(x, y)| \, d\nu(y) < \infty \text{ for almost every } x \in X$$

and

$$\int_X |f(x, y)| \, d\mu(x) < \infty \text{ for almost every } y \in Y.$$

Furthermore,

(a) $$x \mapsto \int_Y f(x, y) \, d\nu(y) \text{ is an } \mathcal{S}\text{-measurable function on } X,$$

(b) $$y \mapsto \int_X f(x, y) \, d\mu(x) \text{ is a } \mathcal{T}\text{-measurable function on } Y,$$

and

$$\int_{X \times Y} f \, d(\mu \times \nu) = \int_X \int_Y f(x, y) \, d\nu(y) \, d\mu(x) = \int_Y \int_X f(x, y) \, d\mu(x) \, d\nu(y).$$

Proof Tonelli's Theorem (5.28) applied to the nonnegative function $|f|$ implies that $x \mapsto \int_Y |f(x, y)| \, d\nu(y)$ is an $\mathcal{S}$-measurable function on X. Hence

$$\left\{ x \in X : \int_Y |f(x, y)| \, d\nu(y) = \infty \right\} \in \mathcal{S}.$$

Tonelli's Theorem applied to $|f|$ also tells us that

$$\int_X \int_Y |f(x, y)| \, d\nu(y) \, d\mu(x) < \infty$$

because the iterated integral above equals $\int_{X \times Y} |f| \, d(\mu \times \nu)$. The inequality above implies that

$$\mu\left(\left\{ x \in X : \int_Y |f(x, y)| \, d\nu(y) = \infty \right\} \right) = 0.$$

Recall that f^+ and f^- are nonnegative $\mathcal{S} \otimes \mathcal{T}$-measurable functions such that $|f| = f^+ + f^-$ and $f = f^+ - f^-$ (see 3.17). Applying Tonelli's Theorem to f^+ and f^-, we see that

5.33 $$x \mapsto \int_Y f^+(x, y) \, d\nu(y) \quad \text{and} \quad x \mapsto \int_Y f^-(x, y) \, d\nu(y)$$

are $\mathcal{S}$-measurable functions from X to $[0,\infty]$. Because $f^+ \le |f|$ and $f^- \le |f|$, the sets $\{x \in X : \int_Y f^+(x,y)\,d\nu(y) = \infty\}$ and $\{x \in X : \int_Y f^-(x,y)\,d\nu(y) = \infty\}$ have μ-measure 0. Thus the intersection of these two sets, which is the set of $x \in X$ such that $\int_Y f(x,y)\,d\nu(y)$ is not defined, also has μ-measure 0.

Subtracting the second function in 5.33 from the first equation in 5.33, we see that the function that we define to be 0 for those $x \in X$ where we encounter $\infty - \infty$ (a set of μ-measure 0, as noted above) and that equals $\int_Y f(x,y)\,d\nu(y)$ elsewhere is an $\mathcal{S}$-measurable function on X.

Now

$$\begin{aligned}
\int_{X\times Y} f\,d(\mu\times\nu) &= \int_{X\times Y} f^+\,d(\mu\times\nu) - \int_{X\times Y} f^-\,d(\mu\times\nu)\\
&= \int_X \int_Y f^+(x,y)\,d\nu(y)\,d\mu(x) - \int_X \int_Y f^-(x,y)\,d\nu(y)\,d\mu(x)\\
&= \int_X \int_Y \bigl(f^+(x,y) - f^-(x,y)\bigr)\,d\nu(y)\,d\mu(x)\\
&= \int_X \int_Y f(x,y)\,d\nu(y)\,d\mu(x),
\end{aligned}$$

where the first line above comes from the definition of the integral of a function that is not nonnegative (note that neither of the two terms on the right side of the first line equals ∞ because $\int_{X\times Y} |f|\,d(\mu\times\nu) < \infty$) and the second line comes applying Tonelli's Theorem to f^+ and f^-.

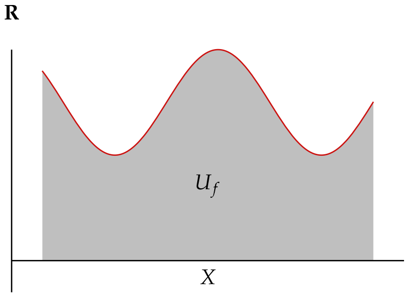
We have now proved all aspects of Fubini's Theorem that involve integrating first over Y. The same procedure provides proofs for the aspects of Fubini's theorem that involve integrating first over X. ■

Area Under the Graph of a Function

5.34 Definition ***region under the graph***

Suppose X is a set and $f\colon X \to [0,\infty]$ is a function. Then the *region under the graph* of f, denoted U_f, is defined by

$$U_f = \{(x,t) \in X \times (0,\infty) : 0 < t < f(x)\}.$$



The figure indicates why we call U_f the region under the graph of f, even in cases when X is not a subset of $\mathbf{R}$. Similarly, the informal term *area* in the next paragraph should remind you of the area in the figure, even though we are really dealing with the measure of U_f in a product space.

The first equality in the result below can be thought of as recovering Riemann's conception of the integral as the area under the graph (although now in a much more general context with arbitrary σ-finite measures). The second equality in the result below can be thought of as reinforcing Lebesgue's conception of computing the area under a curve by integrating in the direction perpendicular to Riemann's.

5.35 *Area under the graph of a function equals the integral*

Suppose $(X, \mathcal{S}, \mu)$ is a σ-finite measure space and $f\colon X \to [0, \infty]$ is an $\mathcal{S}$-measurable function. Let $\mathcal{B}$ denote the σ-algebra of Borel subsets of $(0, \infty)$, and let λ denote Lebesgue measure on $\big((0, \infty), \mathcal{B}\big)$. Then $U_f \in \mathcal{S} \otimes \mathcal{B}$ and

$$(\mu \times \lambda)(U_f) = \int_X f \, d\mu = \int_{(0,\infty)} \mu(\{x \in X : f(x) > t\}) \, d\lambda(t).$$

Proof For $k \in \mathbf{Z}^+$, let

$$E_k = \bigcup_{m=0}^{k^2-1} \Big(f^{-1}\big([\tfrac{m}{k}, \tfrac{m+1}{k})\big) \times \big(0, \tfrac{m}{k}\big) \Big) \quad \text{and} \quad F_k = f^{-1}([k, \infty]) \times (0, k).$$

Then E_k is a finite union of $\mathcal{S} \otimes \mathcal{B}$-measurable rectangles and F_k is an $\mathcal{S} \otimes \mathcal{B}$-measurable rectangle. Because

$$U_f = \bigcup_{k=1}^{\infty} (E_k \cup F_k),$$

we conclude that $U_f \in \mathcal{S} \otimes \mathcal{B}$.

Now the definition of the product measure $\mu \times \lambda$ implies that

$$\begin{aligned}(\mu \times \lambda)(U_f) &= \int_X \int_{(0,\infty)} \chi_{U_f}(x, t) \, d\lambda(t) \, d\mu(x) \\ &= \int_X f(x) \, d\mu(x),\end{aligned}$$

which completes the proof of the first equality in the conclusion of this theorem.

Tonelli's Theorem (5.28) tells us that we can interchange the order of integration in the double integral above, getting

$$\begin{aligned}(\mu \times \lambda)(U_f) &= \int_{(0,\infty)} \int_X \chi_{U_f}(x, t) \, d\mu(x) \, d\lambda(t) \\ &= \int_{(0,\infty)} \mu(\{x \in X : f(x) > t\}) \, d\lambda(t),\end{aligned}$$

which completes the proof of the second equality in the conclusion of this theorem. ∎

Markov's Inequality (4.1) implies that if f and μ are as in the result above, then

$$\mu(\{x \in X : f(x) > t\}) \leq \frac{\int_X f \, d\mu}{t}$$

for all $t > 0$. Thus if $\int_X f \, d\mu < \infty$, then the result above should be considered to be somewhat stronger than Markov's inequality (because $\int_{(0,\infty)} \frac{1}{t} \, d\lambda(t) = \infty$).

EXERCISES 5B

1 (a) Let λ denote Lebesgue measure on $[0,1]$. Show that

$$\int_{[0,1]} \int_{[0,1]} \frac{x^2 - y^2}{(x^2 + y^2)^2} \, d\lambda(y) \, d\lambda(x) = \frac{\pi}{4}$$

and

$$\int_{[0,1]} \int_{[0,1]} \frac{x^2 - y^2}{(x^2 + y^2)^2} \, d\lambda(x) \, d\lambda(y) = -\frac{\pi}{4}.$$

(b) Explain why (a) violates neither Tonelli's Theorem nor Fubini's Theorem.

2 (a) Give an example of a doubly-indexed collection $\{x_{m,n} : m, n \in \mathbf{Z}^+\}$ of real numbers such that

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} x_{m,n} = 0 \quad \text{and} \quad \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} x_{m,n} = \infty.$$

(b) Explain why (a) violates neither Tonelli's Theorem nor Fubini's Theorem.

3 Suppose $(X, \mathcal{S})$ is a measurable space and $f \colon X \to [0, \infty]$ is a function. Let $\mathcal{B}$ denote the σ-algebra of Borel subsets of $(0, \infty)$. Prove that $U_f \in \mathcal{S} \otimes \mathcal{B}$ if and only if f is an $\mathcal{S}$-measurable function.

4 Suppose $(X, \mathcal{S})$ is a measurable space and $f \colon X \to \mathbf{R}$ is a function. Let $\operatorname{graph}(f) \subset X \times \mathbf{R}$ denote the graph of f:

$$\operatorname{graph}(f) = \{(x, f(x)) : x \in X\}.$$

Let $\mathcal{B}$ denote the σ-algebra of Borel subsets of $\mathbf{R}$. Prove that $\operatorname{graph}(f) \in \mathcal{S} \otimes \mathcal{B}$ if and only if f is an $\mathcal{S}$-measurable function.

5C Lebesgue Integration on $\mathbf{R}^n$

Throughout this section, assume that m and n are positive integers. Thus, for example, 5.36 should include the hypothesis that m and n are positive integers, but theorems and definitions become easier to state without explicitly repeating this hypothesis.

Borel Subsets of $\mathbf{R}^n$

We begin with a quick review of notation and key concepts concerning $\mathbf{R}^n$. See Sections D and E of the Appendix for more details and results about $\mathbf{R}^n$.

Recall that $\mathbf{R}^n$ is the set of all n-tuples of real numbers:

$$\mathbf{R}^n = \{(x_1, \dots, x_n) : x_1, \dots, x_n \in \mathbf{R}\}.$$

The function $\|\cdot\|_\infty$ from $\mathbf{R}^n$ to $[0, \infty)$ is defined by

$$\|(x_1, \dots, x_n)\|_\infty = \max\{|x_1|, \dots, |x_n|\}.$$

For $x \in \mathbf{R}^n$ and $\delta > 0$, the *open cube* $B(x, \delta)$ with side length 2δ is defined by

$$B(x, \delta) = \{y \in \mathbf{R}^n : \|y - x\|_\infty < \delta\}.$$

If $n = 1$, then an open cube is simply a bounded open interval. If $n = 2$, then an open cube might more appropriately be called an open square. However, using the cube terminology for all dimensions has the advantage of not requiring a different word for different dimensions.

A subset G of $\mathbf{R}^n$ is called *open* if for every $x \in G$, there exists $\delta > 0$ such that $B(x, \delta) \subset G$. Equivalently, a subset G of $\mathbf{R}^n$ is called *open* if every element of G is contained in an open cube that is contained in G.

The union of every collection of open subsets of $\mathbf{R}^n$ is an open subset of $\mathbf{R}^n$; also, the intersection of every finite collection of open subsets of $\mathbf{R}^n$ is an open subset of $\mathbf{R}^n$ (see 0.55 in the Appendix).

A subset of $\mathbf{R}^n$ is called *closed* if its complement in $\mathbf{R}^n$ is open. A set $A \subset \mathbf{R}^n$ is called *bounded* if $\sup\{\|a\|_\infty : a \in A\} < \infty$.

We adopt the following common convention:

$$\mathbf{R}^m \times \mathbf{R}^n \text{ is identified with } \mathbf{R}^{m+n}.$$

To understand the necessity of this convention, note that $\mathbf{R}^2 \times \mathbf{R} \neq \mathbf{R}^3$ because $\mathbf{R}^2 \times \mathbf{R}$ and $\mathbf{R}^3$ contain different kinds of objects. Specifically, an element of $\mathbf{R}^2 \times \mathbf{R}$ is an ordered pair, the first of which is an element of $\mathbf{R}^2$ and the second of which is an element of $\mathbf{R}$; thus an element of $\mathbf{R}^2 \times \mathbf{R}$ looks like $\big((x_1, x_2), x_3\big)$. An element of $\mathbf{R}^3$ is an ordered triple of real numbers that looks like (x_1, x_2, x_3). However, we can identify $\big((x_1, x_2), x_3\big)$ with (x_1, x_2, x_3) in the obvious way. Thus we say that $\mathbf{R}^2 \times \mathbf{R}$ "equals" $\mathbf{R}^3$. More generally, we make the natural identification of $\mathbf{R}^m \times \mathbf{R}^n$ with $\mathbf{R}^{m+n}$.

To check that you understand the identification discussed above, make sure that you see why $B(x, \delta) \times B(y, \delta) = B\big((x, y), \delta\big)$ for all $x \in \mathbf{R}^m$, $y \in \mathbf{R}^n$, and $\delta > 0$.

We can now prove that the product of two open sets is an open set.

5.36 *Product of open sets is open*

Suppose G_1 is an open subset of $\mathbf{R}^m$ and G_2 is an open subset of $\mathbf{R}^n$. Then $G_1 \times G_2$ is an open subset of $\mathbf{R}^{m+n}$.

Proof Suppose $(x, y) \in G_1 \times G_2$. Then there exists an open cube D in $\mathbf{R}^m$ centered at x and an open cube E in $\mathbf{R}^n$ centered at y such that $D \subset G_1$ and $E \subset G_2$. By reducing the size of either D or E, we can assume that the cubes D and E have the same side length. Thus $D \times E$ is an open cube in $\mathbf{R}^{m+n}$ centered at (x, y) that is contained in $G_1 \times G_2$.

We have shown that an arbitrary point in $G_1 \times G_2$ is the center of an open cube contained in $G_1 \times G_2$. Hence $G_1 \times G_2$ is an open subset of $\mathbf{R}^{m+n}$. ■

When $n = 1$, the definition below of a Borel subset of $\mathbf{R}^1$ agrees with our previous definition (2.28) of a Borel subset of $\mathbf{R}$.

5.37 Definition *Borel set; $\mathcal{B}_n$*

- A *Borel subset* of $\mathbf{R}^n$ is an element of the smallest σ-algebra on $\mathbf{R}^n$ containing all open subsets of $\mathbf{R}^n$.
- The σ-algebra of Borel subsets of $\mathbf{R}^n$ is denoted by $\mathcal{B}_n$.

Recall that a subset of $\mathbf{R}$ is open if and only if it is a countable disjoint union of open intervals (see 0.59 in the Appendix). Part (a) in the result below provides a similar result in $\mathbf{R}^n$, although we must give up the disjoint aspect.

5.38 *Open sets are countable unions of open cubes*

(a) A subset of $\mathbf{R}^n$ is open in $\mathbf{R}^n$ if and only if it is a countable union of open cubes in $\mathbf{R}^n$.

(b) $\mathcal{B}_n$ is the smallest σ-algebra on $\mathbf{R}^n$ containing all the open cubes in $\mathbf{R}^n$.

Proof We will prove (a), which clearly implies (b).

The proof that a countable union of open cubes is open is left as an exercise for the reader (actually, arbitrary unions of open cubes are open).

To prove the other direction, suppose G is an open subset of $\mathbf{R}^n$. For each $x \in G$, there is an open cube centered at x that is contained in G. Thus there is a smaller cube C_x such that $x \in C_x \subset G$ and all coordinates of the center of C_x are rational numbers and the side length of C_x is a rational number. Now

$$G = \bigcup_{x \in G} C_x.$$

However, there are only countably many distinct cubes whose center has all rational coordinates and whose side length is rational. Thus G is the countable union of open cubes. ■

The next result tells us that the collection of Borel sets from various dimensions fit together nicely.

5.39 ***The product of the Borel subsets of*** $\mathbf{R}^m$ ***and the Borel subsets of*** $\mathbf{R}^n$

$\mathcal{B}_m \otimes \mathcal{B}_n = \mathcal{B}_{m+n}$.

Proof Suppose E is an open cube in $\mathbf{R}^{m+n}$. Thus E is the product of an open cube in $\mathbf{R}^m$ and an open cube in $\mathbf{R}^n$. Hence $E \in \mathcal{B}_m \otimes \mathcal{B}_n$. Thus the smallest σ-algebra containing all the open cubes in $\mathbf{R}^{m+n}$ is contained in $\mathcal{B}_m \otimes \mathcal{B}_n$. Now 5.38(b) implies that $\mathcal{B}_{m+n} \subset \mathcal{B}_m \otimes \mathcal{B}_n$.

To prove a set inclusion in the other direction, temporarily fix an open set G in $\mathbf{R}^n$. Let

$$\mathcal{E} = \{A \subset \mathbf{R}^m : A \times G \in \mathcal{B}_{m+n}\}.$$

Then $\mathcal{E}$ contains every open subset of $\mathbf{R}^m$ (as follows from 5.36). Also, $\mathcal{E}$ is closed under countable unions because

$$\Big(\bigcup_{k=1}^{\infty} A_k\Big) \times G = \bigcup_{k=1}^{\infty} (A_k \times G).$$

Furthermore, $\mathcal{E}$ is closed under complementation because

$$\big(\mathbf{R}^m \setminus A\big) \times G = \Big(\big(\mathbf{R}^m \times \mathbf{R}^n\big) \setminus \big(A \times G\big)\Big) \cap \big(\mathbf{R}^m \times G\big).$$

Thus $\mathcal{E}$ is a σ-algebra on $\mathbf{R}^m$ that contains all open subsets of $\mathbf{R}^m$, which implies that $\mathcal{B}_m \subset \mathcal{E}$. In other words, we have proved that if $A \in \mathcal{B}_m$ and G is an open subset of $\mathbf{R}^n$, then $A \times G \in \mathcal{B}_{m+n}$.

Now temporarily fix a Borel subset A of $\mathbf{R}^m$. Let

$$\mathcal{F} = \{B \subset \mathbf{R}^n : A \times B \in \mathcal{B}_{m+n}\}.$$

The conclusion of the previous paragraph shows that $\mathcal{F}$ contains every open subset of $\mathbf{R}^n$. As in the previous paragraph, we also see that $\mathcal{F}$ is a σ-algebra. Hence $\mathcal{B}_n \subset \mathcal{F}$. In other words, we have proved that if $A \in \mathcal{B}_m$ and $B \in \mathcal{B}_n$, then $A \times B \in \mathcal{B}_{m+n}$. Thus $\mathcal{B}_m \otimes \mathcal{B}_n \subset \mathcal{B}_{m+n}$, completing the proof. ■

The previous result implies a nice associative property. Specifically, if m, n, and p are positive integers, then two applications of 5.39 give

$$(\mathcal{B}_m \otimes \mathcal{B}_n) \otimes \mathcal{B}_p = \mathcal{B}_{m+n} \otimes \mathcal{B}_p = \mathcal{B}_{m+n+p}.$$

Similarly, two more applications of 5.39 give

$$\mathcal{B}_m \otimes (\mathcal{B}_n \otimes \mathcal{B}_p) = \mathcal{B}_m \otimes \mathcal{B}_{n+p} = \mathcal{B}_{m+n+p}.$$

Thus $(\mathcal{B}_m \otimes \mathcal{B}_n) \otimes \mathcal{B}_p = \mathcal{B}_m \otimes (\mathcal{B}_n \otimes \mathcal{B}_p)$; hence we can dispense with parentheses when taking products of more than two Borel σ-algebras. More generally, we could have defined $\mathcal{B}_m \otimes \mathcal{B}_n \otimes \mathcal{B}_p$ directly as the smallest σ-algebra on $\mathbf{R}^{m+n+p}$ containing $\{A \times B \times C : A \in \mathcal{B}_m, B \in \mathcal{B}_n, C \in \mathcal{B}_p\}$ and obtained the same σ-algebra (see Exercise 3 in this section).

Lebesgue Measure on $\mathbf{R}^n$

5.40 Definition ***Lebesgue measure;*** λ_n

Lebesgue measure on $\mathbf{R}^n$ is denoted by λ_n is defined inductively by

$$\lambda_n = \lambda_{n-1} \times \lambda_1,$$

where λ_1 is Lebesgue measure on $(\mathbf{R}, \mathcal{B}_1)$.

Because $\mathcal{B}_n = \mathcal{B}_{n-1} \otimes \mathcal{B}_1$ (by 5.39), the measure λ_n is defined on the Borel subsets of $\mathbf{R}^n$. Thinking of a typical point in $\mathbf{R}^n$ as (x, y), where $x \in \mathbf{R}^{n-1}$ and $y \in \mathbf{R}$, we can use the definition of the product of two measures (5.25) to write

$$\lambda_n(E) = \int_{\mathbf{R}^{n-1}} \int_{\mathbf{R}} \chi_E(x, y) \, d\lambda_1(y) \, d\lambda_{n-1}(x)$$

for $E \in \mathcal{B}_n$. Of course, we could use Tonelli's Theorem (5.28) to interchange the order of integration in the equation above.

Because Lebesgue measure is the most commonly used measure, mathematicians often dispense with explicitly displaying the measure and just use a variable name. In other words, if no measure is explicitly displayed in an integral and the context indicates no other measure, then you should assume that the measure involved is Lebesgue measure in the appropriate dimension. For example, the result of interchanging the order of integration in the equation above could be written as

$$\lambda_n(E) = \int_{\mathbf{R}} \int_{\mathbf{R}^{n-1}} \chi_E(x, y) \, dx \, dy$$

for $E \in \mathcal{B}_n$; here dx means $d\lambda_{n-1}(x)$ and dy means $d\lambda_1(y)$.

In the equations above giving formulas for $\lambda_n(E)$, the integral over $\mathbf{R}^{n-1}$ could be rewritten as an iterated integral over $\mathbf{R}^{n-2}$ and $\mathbf{R}$, and that process could be repeated until reaching iterated integrals only over $\mathbf{R}$. Tonelli's Theorem could then be used repeatedly to swap the order of pairs of those integrated integrals, leading to iterated integrals in any order.

Similar comments apply to integrating functions on $\mathbf{R}^n$ other than characteristic functions. For example, if $f \colon \mathbf{R}^3 \to \mathbf{R}$ is a $\mathcal{B}_3$-measurable function such that either $f \geq 0$ or $\int_{\mathbf{R}^3} |f| \, d\lambda_3 < \infty$, then by either Tonelli's Theorem or Fubini's Theorem we have

$$\int_{\mathbf{R}^3} f \, d\lambda_3 = \int_{\mathbf{R}} \int_{\mathbf{R}} \int_{\mathbf{R}} f(x_1, x_2, x_3) \, dx_j \, dx_k \, dx_m,$$

where j, k, m is any permutation of $1, 2, 3$.

Although we defined λ_n to be $\lambda_{n-1} \times \lambda_1$, we could have defined λ_n to be $\lambda_j \times \lambda_k$ for any positive integers j, k with $j + k = n$. This potentially different definition would have led to the same σ-algebra $\mathcal{B}_n$ (by 5.39) and to the same measure λ_n [because both potential definitions of $\lambda_n(E)$ can be written as identical iterations of n integrals with respect to λ_1].

Volume of the Unit Ball in $\mathbf{R}^n$

The proof of the next result provides good experience in working with the Lebesgue measure λ_n. Recall that $tE = \{tx : x \in E\}$.

5.41 ***measure of a dilation***

Suppose $t > 0$. If $E \in \mathcal{B}_n$, then $tE \in \mathcal{B}_n$ and $\lambda_n(tE) = t^n \lambda_n(E)$.

Proof Let

$$\mathcal{E} = \{E \in \mathcal{B}_n : tE \in \mathcal{B}_n\}.$$

Then $\mathcal{E}$ contains every open subset of $\mathbf{R}^n$ (because if E is open in $\mathbf{R}^n$ then tE is open in $\mathbf{R}^n$). Also, $\mathcal{E}$ is closed under complementation and countable unions because

$$t(\mathbf{R}^n \setminus E) = \mathbf{R}^n \setminus (tE) \quad \text{and} \quad t\Bigl(\bigcup_{k=1}^{\infty} E_k\Bigr) = \bigcup_{k=1}^{\infty} (tE_k).$$

Hence $\mathcal{E}$ is a σ-algebra on $\mathbf{R}^n$ containing the open subsets of $\mathbf{R}^n$. Thus $\mathcal{E} = \mathcal{B}_n$. In other words, $tE \in \mathcal{B}_n$ for all $E \in \mathcal{B}_n$.

To prove $\lambda_n(tE) = t^n \lambda_n(E)$, first consider the case $n = 1$. Lebesgue measure on $\mathbf{R}$ is a restriction of outer measure. The outer measure of a set is determined by the sum of the lengths of countable collections of intervals whose union contains the set. Multiplying the set by t corresponds to multiplying each such interval by t, which multiplies the length of each such interval by t. In other words, $\lambda_1(tE) = t\lambda_1(E)$.

Now assume $n > 1$. We will use induction on n and assume that the desired result holds for $n - 1$. If $A \in \mathcal{B}_{n-1}$ and $B \in \mathcal{B}_1$, then

$$\begin{aligned}
\lambda_n\bigl(t(A \times B)\bigr) &= \lambda_n\bigl((tA) \times (tB)\bigr) \\
&= \lambda_{n-1}(tA) \cdot \lambda_1(tB) \\
&= t^{n-1}\lambda_{n-1}(A) \cdot t\lambda_1(B) \\
&= t^n \lambda_n(A \times B),
\end{aligned} \tag{5.42}$$

giving the desired result for $A \times B$.

For $m \in \mathbf{Z}^+$, let C_m be the open cube in $\mathbf{R}^n$ centered at the origin and with side length m. Let

$$\mathcal{E}_m = \{E \in \mathcal{B}_n : E \subset C_m \text{ and } \lambda_n(tE) = t^n \lambda_n(E)\}.$$

From 5.42 and using 5.13(b), we see that finite unions of measurable rectangles contained in C_m are in $\mathcal{E}_m$. You should verify that $\mathcal{E}_m$ is closed under countable increasing unions (use 2.58) and countable decreasing intersections (use 2.59, whose finite measure condition holds because we are working inside C_m). From 5.13 and the Monotone Class Theorem (5.17), we conclude that $\mathcal{E}_m$ is the σ-algebra on C_m consisting of Borel subsets of C_m. Thus $\lambda_n(tE) = t^n \lambda_n(E)$ for all $E \in \mathcal{B}_n$ such that $E \subset C_m$.

Now suppose $E \in \mathcal{B}_n$. Then 2.58 implies that

$$\lambda_n(tE) = \lim_{m\to\infty} \lambda_n\bigl(t(E \cap C_m)\bigr) = t^n \lim_{m\to\infty} \lambda_n(E \cap C_m) = t^n \lambda_n(E),$$

as desired. ∎

5.43 Definition ***open unit ball in $\mathbf{R}^n$***

The *open unit ball* in $\mathbf{R}^n$ is denoted by $\mathbf{B}_n$ and is defined by

$$\mathbf{B}_n = \{(x_1, \ldots, x_n) \in \mathbf{R}^n : {x_1}^2 + \cdots + {x_n}^2 < 1\}.$$

The open unit ball $\mathbf{B}_n$ is open in $\mathbf{R}^n$ (as you should verify) and thus is in the collection $\mathcal{B}_n$ of Borel sets.

5.44 ***Volume of the unit ball in $\mathbf{R}^n$***

$$\lambda_n(\mathbf{B}_n) = \begin{cases} \dfrac{\pi^{n/2}}{(n/2)!} & \text{if } n \text{ is even,} \\[2ex] \dfrac{2^{(n+1)/2}\,\pi^{(n-1)/2}}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot n} & \text{if } n \text{ is odd.} \end{cases}$$

Proof Because $\lambda_1(\mathbf{B}_1) = 2$ and $\lambda_2(\mathbf{B}_2) = \pi$, the claimed formula is correct when $n = 1$ and when $n = 2$.

Now assume that $n > 2$. We will use induction on n, assuming that the claimed formula is true for smaller values of n. Think of $\mathbf{R}^n = \mathbf{R}^2 \times \mathbf{R}^{n-2}$ and $\lambda_n = \lambda_2 \times \lambda_{n-2}$. Then

5.45
$$\lambda_n(\mathbf{B}_n) = \int_{\mathbf{R}^2} \int_{\mathbf{R}^{n-2}} \chi_{\mathbf{B}_n}(x, y)\, dy\, dx.$$

Temporarily fix $x = (x_1, x_2) \in \mathbf{R}^2$. If ${x_1}^2 + {x_2}^2 \geq 1$, then $\chi_{\mathbf{B}_n}(x, y) = 0$ for all $y \in \mathbf{R}^{n-2}$. If ${x_1}^2 + {x_2}^2 < 1$ and $y \in \mathbf{R}^{n-2}$, then $\chi_{\mathbf{B}_n}(x, y) = 1$ if and only if $y \in (1 - {x_1}^2 - {x_2}^2)^{1/2}\mathbf{B}_{n-2}$. Thus the inner integral in 5.45 equals

$$\lambda_{n-2}\Big((1 - {x_1}^2 - {x_2}^2)^{1/2}\mathbf{B}_{n-2}\Big)\chi_{\mathbf{B}_2}(x),$$

which by 5.41 equals

$$(1 - {x_1}^2 - {x_2}^2)^{(n-2)/2}\lambda_{n-2}(\mathbf{B}_{n-2})\chi_{\mathbf{B}_2}(x).$$

Thus 5.45 becomes the equation

$$\lambda_n(\mathbf{B}_n) = \lambda_{n-2}(\mathbf{B}_{n-2}) \int_{\mathbf{B}_2} (1 - {x_1}^2 - {x_2}^2)^{(n-2)/2}\, d\lambda_2(x_1, x_2).$$

To evaluate this integral, switch to the usual polar coordinates that you learned about in calculus ($d\lambda_2 = r\, dr\, d\theta$), getting

$$\begin{aligned} \lambda_n(\mathbf{B}_n) &= \lambda_{n-2}(\mathbf{B}_{n-2}) \int_{-\pi}^{\pi} \int_0^1 (1 - r^2)^{(n-2)/2} r\, dr\, d\theta \\ &= \frac{2\pi}{n}\lambda_{n-2}(\mathbf{B}_{n-2}). \end{aligned}$$

The last equation and the induction hypothesis give the desired result. ∎

n	$\lambda_n(\mathbf{B}_n)$	$\approx \lambda_n(\mathbf{B}_n)$
1	2	2.00
2	π	3.14
3	$4\pi/3$	4.19
4	$\pi^2/2$	4.93
5	$8\pi^2/15$	5.26

The table here gives the first five values of $\lambda_n(\mathbf{B}_n)$, using 5.44. The last column of this table gives a decimal approximation to $\lambda_n(\mathbf{B}_n)$, accurate to two digits after the decimal point. From this table, you might guess that $\lambda_n(\mathbf{B}_n)$ is an increasing function of n, especially because the smallest cube containing the ball $\mathbf{B}_n$ has n-dimensional Lebesgue measure 2^n. However, Exercise 11 in this section shows that $\lambda_n(\mathbf{B}_n)$ behaves much differently.

Equality of Mixed Partial Derivatives Via Fubini's Theorem

5.46 Definition ***partial derivatives; $D_1 f$ and $D_2 f$***

Suppose G is an open subset of $\mathbf{R}^2$ and $f\colon G \to \mathbf{R}$ is a function. For $(x, y) \in G$, the *partial derivatives* $(D_1 f)(x, y)$ and $(D_2 f)(x, y)$ are defined by

$$(D_1 f)(x, y) = \lim_{t \to 0} \frac{f(x+t, y) - f(x, y)}{t}$$

and

$$(D_2 f)(x, y) = \lim_{t \to 0} \frac{f(x, y+t) - f(x, y)}{t}$$

if these limits exist.

Using the notation for the cross section of a function (see 5.7), we could write the definitions of D_1 and D_2 in the following form:

$$(D_1 f)(x, y) = ([f]^y)'(x) \quad \text{and} \quad (D_2 f)(x, y) = ([f]_x)'(y).$$

5.47 Example *partial derivatives of x^y*

Let $G = \{(x, y) \in \mathbf{R}^2 : x > 0\}$ and define $f\colon G \to \mathbf{R}$ by $f(x, y) = x^y$. Then

$$(D_1 f)(x, y) = yx^{y-1} \quad \text{and} \quad (D_2 f)(x, y) = x^y \ln x,$$

as you should verify. Taking partial derivatives of those partial derivatives, we have

$$\bigl(D_2(D_1 f)\bigr)(x, y) = x^{y-1} + yx^{y-1} \ln x$$

and

$$\bigl(D_1(D_2 f)\bigr)(x, y) = x^{y-1} + yx^{y-1} \ln x,$$

as you should also verify. The last two equations show that $D_1(D_2 f) = D_2(D_1 f)$ as functions on G.

In the example above, the two mixed partial derivatives turn out to equal to each other, even though the intermediate results look quite different. The next result shows that the behavior in the example above is typical rather than a coincidence.

Some proofs of the result below do not use Fubini's Theorem. However, Fubini's Theorem leads to the clean proof below.

There exist versions of the result below with slightly different hypotheses, but the hypotheses used here are usually easy to verify in practice. Although the continuity hypotheses used here can be slightly weakened, they cannot be eliminated, as shown by Exercise 14 in this section.

The integrals in the proof below make sense because continuous real-valued functions on $\mathbf{R}^2$ are measurable (because the inverse image of each open set is open; see Exercise 18 in the Appendix) and continuous real-valued functions on closed bounded subsets of $\mathbf{R}^2$ are bounded (see 0.80 in the Appendix).

5.48 *Equality of mixed partial derivatives*

Suppose G is an open subset of $\mathbf{R}^2$ and $f\colon G \to \mathbf{R}$ is a function such that $D_1 f$, $D_2 f$, $D_1(D_2 f)$, and $D_2(D_1 f)$ all exist and are continuous functions on G. Then

$$D_1(D_2 f) = D_2(D_1 f)$$

on G.

Proof Fix $(a,b) \in G$. For $\delta > 0$, let $S_\delta = [a, a+\delta] \times [b, b+\delta]$. If $S_\delta \subset G$, then

$$\begin{aligned}
\int_{S_\delta} D_1(D_2 f)\, d\lambda_2 &= \int_b^{b+\delta} \int_a^{a+\delta} \big(D_1(D_2 f)\big)(x,y)\, dx\, dy \\
&= \int_b^{b+\delta} \big[(D_2 f)(a+\delta, y) - (D_2 f)(a, y)\big]\, dy \\
&= f(a+\delta, b+\delta) - f(a+\delta, b) - f(a, b+\delta) + f(a,b),
\end{aligned}$$

where the first equality comes from Fubini's Theorem (5.32) and the second and third equalities come from the Fundamental Theorem of Calculus.

A similar calculation of $\int_{S_\delta} D_2(D_1 f)\, d\lambda_2$ yields the same result. Thus

$$\int_{S_\delta} [D_1(D_2 f) - D_2(D_1 f)]\, d\lambda_2 = 0$$

for all δ such that $S_\delta \subset G$. If $\big(D_1(D_2 f)\big)(a,b) > \big(D_2(D_1 f)\big)(a,b)$, then by the continuity of $D_1(D_2 f)$ and $D_2(D_1 f)$, the integrand in the equation above is positive on S_δ for δ sufficiently small, which contradicts the integral above equaling 0. Similarly, the inequality $\big(D_1(D_2 f)\big)(a,b) < \big(D_2(D_1 f)\big)(a,b)$ also contradicts the equation above for small δ. Thus we conclude that $\big(D_1(D_2 f)\big)(a,b) = \big(D_2(D_1 f)\big)(a,b)$, as desired. ■

EXERCISES 5C

1 Show that a set $G \subset \mathbf{R}^n$ is open in $\mathbf{R}^n$ if and only if for each $(b_1, \dots, b_n) \in G$, there exists $r > 0$ such that

$$\Big\{(a_1, \dots, a_n) \in \mathbf{R}^n : \sqrt{(a_1 - b_1)^2 + \cdots + (a_n - b_n)^2} < r\Big\} \subset G.$$

2 Show that there exists a set $E \subset \mathbf{R}^2$ (thinking of $\mathbf{R}^2$ as equal to $\mathbf{R} \times \mathbf{R}$) such that the cross sections $[E]_a$ and $[E]^a$ are open subsets of $\mathbf{R}$ for every $a \in \mathbf{R}$, but $E \notin \mathcal{B}_2$.

3 Suppose $(X, \mathcal{S})$, $(Y, \mathcal{T})$, and $(Z, \mathcal{U})$ are measurable spaces. We can define $\mathcal{S} \otimes \mathcal{T} \otimes \mathcal{U}$ to be the smallest σ-algebra on $X \times Y \times Z$ that contains

$$\{A \times B \times C : A \in \mathcal{S}, B \in \mathcal{T}, C \in \mathcal{U}\}.$$

Prove that if we make the obvious identifications of the products $(X \times Y) \times Z$ and $X \times (Y \times Z)$ with $X \times Y \times Z$, then

$$\mathcal{S} \otimes \mathcal{T} \otimes \mathcal{U} = (\mathcal{S} \otimes \mathcal{T}) \otimes \mathcal{U} = \mathcal{S} \otimes (\mathcal{T} \otimes \mathcal{U}).$$

4 Show that Lebesgue measure on $\mathbf{R}^n$ is translation invariant. More precisely, show that if $E \in \mathcal{B}_n$ and $a \in \mathbf{R}^n$, then $a + E \in \mathcal{B}_n$ and $\lambda_n(a + E) = \lambda_n(E)$, where

$$a + E = \{a + x : x \in E\}.$$

5 Suppose $f \colon \mathbf{R}^n \to \mathbf{R}$ is $\mathcal{B}_n$-measurable and $t > 0$. Define $f_t \colon \mathbf{R}^n \to \mathbf{R}$ by $f_t(x) = f(tx)$.

(a) Prove that f_t is $\mathcal{B}_n$ measurable.

(b) Prove that if $\int_{\mathbf{R}^n} f \, d\lambda_n$ is defined, then

$$\int_{\mathbf{R}^n} f_t \, d\lambda_n = \frac{1}{t^n} \int_{\mathbf{R}^n} f \, d\lambda_n.$$

6 Suppose λ denotes Lebesgue measure on $(\mathbf{R}, \mathcal{L})$, where $\mathcal{L}$ is the σ-algebra of Lebesgue measurable subsets of $\mathbf{R}$. Show that there exist subsets E and F of $\mathbf{R}^2$ such that

- $F \in \mathcal{L} \otimes \mathcal{L}$ and $(\lambda \times \lambda)(F) = 0$;
- $E \subset F$ but $E \notin \mathcal{L} \otimes \mathcal{L}$.

[*The measure space* $(\mathbf{R}, \mathcal{L}, \lambda)$ *has the property that every subset of a set with measure* 0 *is measurable. This exercise asks you to show that the measure space* $(\mathbf{R}^2, \mathcal{L} \otimes \mathcal{L}, \lambda \times \lambda)$ *does not have this property.*]

7 Suppose $M \in \mathbf{Z}^+$. Verify that the collection of sets $\mathcal{E}_M$ that appears in the proof of 5.41 is a monotone class.

8 Suppose G_1 is a nonempty subset of $\mathbf{R}^m$ and G_2 is a nonempty subset of $\mathbf{R}^n$. Prove that $G_1 \times G_2$ is an open subset of $\mathbf{R}^m \times \mathbf{R}^n$ if and only if G_1 is an open subset of $\mathbf{R}^m$ and G_2 is an open subset of $\mathbf{R}^n$.
[*One direction of this result was already proved* (*see* 5.36); *both directions are stated here to make the result look prettier and to be comparable to the next exercise, where neither direction has been proved.*]

9 Suppose F_1 is a nonempty subset of $\mathbf{R}^m$ and F_2 is a nonempty subset of $\mathbf{R}^n$. Prove that $F_1 \times F_2$ is a closed subset of $\mathbf{R}^m \times \mathbf{R}^n$ if and only if F_1 is a closed subset of $\mathbf{R}^m$ and F_2 is a closed subset of $\mathbf{R}^n$.

10 Show that the open unit ball in $\mathbf{R}^n$ is an open subset of $\mathbf{R}^n$.

11 Prove that $\lim_{n\to\infty} \lambda_n(\mathbf{B}_n) = 0$.

12 Find the value of n that maximizes $\lambda_n(\mathbf{B}_n)$.

13 For readers familiar with the gamma function Γ: Prove that

$$\lambda_n(\mathbf{B}_n) = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}$$

for every positive integer n.

14 Define $f\colon \mathbf{R}^2 \to \mathbf{R}$ by

$$f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

(a) Prove that $D_1(D_2 f)$ and $D_2(D_1 f)$ exist everywhere on $\mathbf{R}^2$.

(b) Show that $\big(D_1(D_2 f)\big)(0,0) \neq \big(D_2(D_1 f)\big)(0,0)$.

(c) Explain why (b) does not violate 5.48.