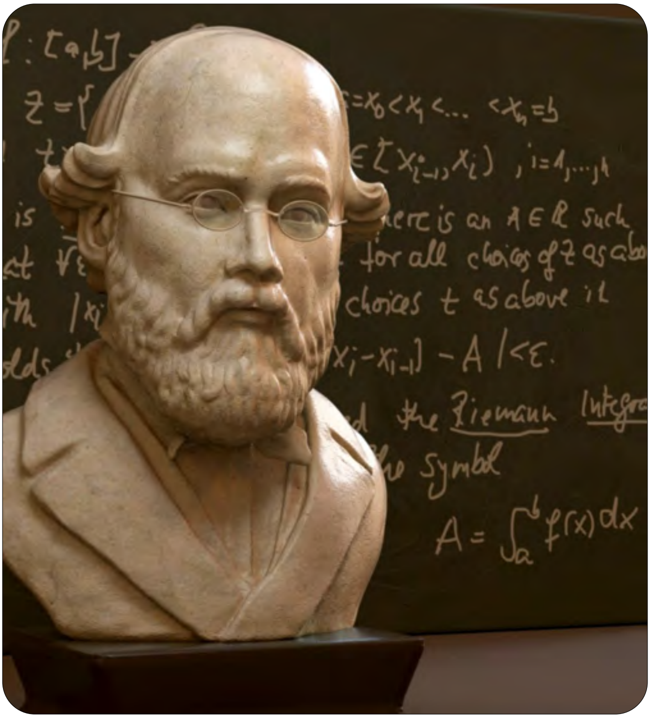
Chapter

1

Riemann Integration

This chapter reviews Riemann integration. Riemann integration uses rectangles to approximate areas under graphs. This chapter begins by carefully presenting the definitions leading to the Riemann integral. The big result in the first section states that a continuous real-valued function on a closed bounded interval is Riemann integrable. The proof depends upon the theorem that continuous functions on closed bounded intervals are uniformly continuous.

The second section of this chapter focuses on several deficiencies of Riemann integration. As we will see, Riemann integration does not do everything that we would like an integral to do. These deficiencies will provide motivation in future chapters for the development of measures and integration with respect to measures.



Digital sculpture of Bernhard Riemann (1826–1866), the German mathematician whose method of integration is taught in calculus courses.

1A Review: Riemann Integral

Let $\mathbf{R}$ denote the complete ordered field of real numbers. See the Appendix for important properties of $\mathbf{R}$, especially related to the concepts of infimum and supremum, that you should know before reading other parts of this book.

1.1 Definition ***partition***

Suppose $a, b \in \mathbf{R}$ with $a < b$. A *partition* of $[a, b]$ is a finite list of the form $x_0, x_1, \dots, x_n$, where

$$a = x_0 < x_1 < \cdots < x_n = b.$$

We use a partition $x_0, x_1, \dots, x_n$ of $[a, b]$ to think of $[a, b]$ as a union of closed subintervals, as follows:

$$[a, b] = [x_0, x_1] \cup [x_1, x_2] \cup \cdots \cup [x_{n-1}, x_n].$$

The next definition introduces clean notation for the infimum and supremum of the values of a function on some subset of its domain.

1.2 Definition ***notation for infimum and supremum of a function***

If f is a real-valued function and A is a subset of the domain of f, then

$$\inf_A f = \inf\{f(x) : x \in A\} \quad \text{and} \quad \sup_A f = \sup\{f(x) : x \in A\}.$$

The lower and upper Riemann sums, which we now define, approximate the area under the graph of a nonnegative function (or, more generally, the signed area corresponding to a real-valued function).

1.3 Definition ***lower and upper Riemann sums***

Suppose $f : [a, b] \to \mathbf{R}$ is a bounded function and P is a partition $x_0, \dots, x_n$ of $[a, b]$. The *lower Riemann sum* $L(f, P, [a, b])$ and the *upper Riemann sum* $U(f, P, [a, b])$ are defined by

$$L(f, P, [a, b]) = \sum_{j=1}^{n} (x_j - x_{j-1}) \inf_{[x_{j-1}, x_j]} f$$

and

$$U(f, P, [a, b]) = \sum_{j=1}^{n} (x_j - x_{j-1}) \sup_{[x_{j-1}, x_j]} f.$$

Our intuition suggests that for a partition with only a small gap between consecutive points, the lower Riemann sum should be a bit less than the area under the graph, and the upper Riemann sum should be a bit more than the area under the graph.

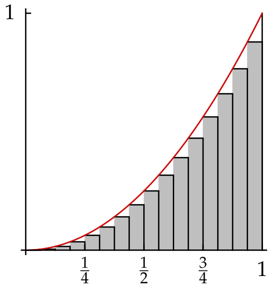
The pictures in the next example help convey the idea of these approximations. The base of the j^{th} rectangle has length $x_j - x_{j-1}$ and has height $\inf\limits_{[x_{j-1},x_j]} f$ for the lower Riemann sum and height $\sup\limits_{[x_{j-1},x_j]} f$ for the upper Riemann sum.

1.4 **Example** *lower and upper Riemann sums*

Define $f\colon [0,1] \to \mathbf{R}$ by

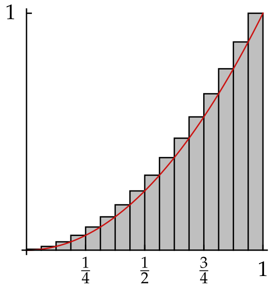
$$f(x) = x^2.$$

Let P_n denote the partition $0, \frac{1}{n}, \frac{2}{n}, \dots, 1$ of $[0,1]$.



$L(x^2, P_{16}, [0,1])$ equals the sum of the areas of these rectangles.

The two figures here show the graph of f in red. The infimum of this function f is attained at the left endpoint of each subinterval $[\frac{j-1}{n}, \frac{j}{n}]$; the supremum is attained at the right endpoint.



$U(x^2, P_{16}, [0,1])$ equals the sum of the areas of these rectangles.

For the partition P_n, we have $x_j - x_{j-1} = \frac{1}{n}$ for each $j = 1, \dots, n$. Thus

$$L(x^2, P_n, [0,1]) = \frac{1}{n} \sum_{j=1}^{n} \frac{(j-1)^2}{n^2} = \frac{2n^2 - 3n + 1}{6n^2}$$

and

$$U(x^2, P_n, [0,1]) = \frac{1}{n} \sum_{j=1}^{n} \frac{j^2}{n^2} = \frac{2n^2 + 3n + 1}{6n^2},$$

as you should verify [use the formula $1 + 4 + 9 + \dots + n^2 = \frac{n(2n^2+3n+1)}{6}$].

The next result states that adjoining more points to a partition increases the lower Riemann sum and decreases the upper Riemann sum.

1.5 *Inequalities with Riemann sums*

Suppose $f\colon [a,b] \to \mathbf{R}$ is a bounded function and P, P' are partitions of $[a,b]$ such that the list defining P is a sublist of the list defining P'. Then

$$L(f, P, [a,b]) \le L(f, P', [a,b]) \le U(f, P', [a,b]) \le U(f, P, [a,b]).$$

Proof To prove the first inequality, suppose P is the partition $x_0, \dots, x_n$ and P' is the partition $x'_0, \dots, x'_N$ of $[a,b]$. For each $j = 1, \dots, n$, there exist $k \in \{0, \dots, N-1\}$ and a positive integer m such that $x_{j-1} = x'_k < x'_{k+1} < \dots < x'_{k+m} = x_j$. We have

$$
\begin{aligned}
(x_j - x_{j-1}) \inf_{[x_{j-1}, x_j]} f &= \sum_{i=1}^{m} (x'_{k+i} - x'_{k+i-1}) \inf_{[x_{j-1}, x_j]} f \\
&\leq \sum_{i=1}^{m} (x'_{k+i} - x'_{k+i-1}) \inf_{[x'_{k+i-1}, x'_{k+i}]} f.
\end{aligned}
$$

The inequality above implies that $L(f, P, [a, b]) \leq L(f, P', [a, b])$.

The middle inequality in this result follows from the observation that the infimum of each set of real numbers is less than or equal to the supremum of that set.

The proof of the last inequality in this result is similar to the proof of the first inequality and is left to the reader. ■

The following result states that if the function is fixed, then each lower Riemann sum is less than or equal to each upper Riemann sum.

1.6 ***Lower Riemann sums $\leq$ upper Riemann sums***

Suppose $f\colon [a, b] \to \mathbf{R}$ is a bounded function and P, P' are partitions of $[a, b]$. Then

$$L(f, P, [a, b]) \leq U(f, P', [a, b]).$$

Proof Let P'' be the partition of $[a, b]$ obtained by merging the lists that define P and P'. Then

$$
\begin{aligned}
L(f, P, [a, b]) &\leq L(f, P'', [a, b]) \\
&\leq U(f, P'', [a, b]) \\
&\leq U(f, P', [a, b]),
\end{aligned}
$$

where all three inequalities above come from 1.5. ■

We have been working with lower and upper Riemann sums. Now we define the lower and upper Riemann integrals.

1.7 **Definition** ***lower and upper Riemann integrals***

Suppose $f\colon [a, b] \to \mathbf{R}$ is a bounded function. The *lower Riemann integral* $L(f, [a, b])$ and the *upper Riemann integral* $U(f, [a, b])$ of f are defined by

$$L(f, [a, b]) = \sup_{P} L(f, P, [a, b])$$

and

$$U(f, [a, b]) = \inf_{P} U(f, P, [a, b]),$$

where the supremum and infimum above are taken over all partitions P of $[a, b]$.

In the definition above, we take the supremum (over all partitions) of the lower Riemann sums because adjoining more points to a partition increases the lower Riemann sum (by 1.5) and should provide a more accurate estimate of the area under the graph. Similarly, in the definition above, we take the infimum (over all partitions) of the upper Riemann sums because adjoining more points to a partition decreases the upper Riemann sum (by 1.5) and should provide a more accurate estimate of the area under the graph.

Our first result about the lower and upper Riemann integrals is an easy inequality.

1.8 *Lower Riemann integral $\leq$ upper Riemann integral*

Suppose $f\colon [a,b] \to \mathbf{R}$ is a bounded function. Then

$$L(f,[a,b]) \leq U(f,[a,b]).$$

Proof The desired inequality follows from the definitions and 1.6. ■

The lower Riemann integral and the upper Riemann integral can both be reasonably considered to be the area under the graph of a function. Which one should we use? The pictures in Example 1.4 suggest that these two quantities are the same for the function in that example; we will soon verify this suspicion. However, as we will see in the next section, there are functions for which the lower Riemann integral does not equal the upper Riemann integral.

Instead of choosing between the lower Riemann integral and the upper Riemann integral, the standard procedure in Riemann integration is to consider only functions for which those two quantities are equal. This decision has the huge advantage of making the Riemann integral behave as we wish with respect to the sum of two functions (see Exercise 5 in this section).

1.9 Definition *Riemann integrable; Riemann integral*

- A bounded function on a closed bounded interval is called *Riemann integrable* if its lower Riemann integral equals its upper Riemann integral.
- If $f\colon [a,b] \to \mathbf{R}$ is Riemann integrable, then the *Riemann integral* $\int_a^b f$ is defined by
$$\int_a^b f = L(f,[a,b]) = U(f,[a,b]).$$

Let $\mathbf{Z}$ denote the set of integers and $\mathbf{Z}^+$ denote the set of positive integers.

1.10 Example *computing a Riemann integral*

Define $f\colon [0,1] \to \mathbf{R}$ by $f(x) = x^2$. Then

$$U(f,[0,1]) \leq \inf_{n\in\mathbf{Z}^+} \frac{2n^2+3n+1}{6n^2} = \frac{1}{3} = \sup_{n\in\mathbf{Z}^+} \frac{2n^2-3n+1}{6n^2} \leq L(f,[0,1]),$$

where the two inequalities above come from Example 1.4 and the two equalities easily follow from dividing the numerators and denominators of both fractions above by n^2.

The paragraph above shows that $U(f,[0,1]) \le \frac{1}{3} \le L(f,[0,1])$. When combined with 1.8, this shows that $L(f,[0,1]) = U(f,[0,1]) = \frac{1}{3}$. Thus f is Riemann integrable and

$$\int_0^1 f = \frac{1}{3}.$$

Now we come to a key result about Riemann integration. Uniform continuity provides the major tool that makes the proof work.

1.11 *Continuous functions are Riemann integrable*

Every continuous real-valued function on each closed bounded interval is Riemann integrable.

Proof Suppose $a, b \in \mathbf{R}$ with $a < b$ and $f : [a,b] \to \mathbf{R}$ is a continuous function (thus f is bounded, by 0.80 in the Appendix). Let $\varepsilon > 0$. Because f is uniformly continuous (by 0.79), there exists $\delta > 0$ such that

1.12 $$|f(s) - f(t)| < \varepsilon \text{ for all } s, t \in [a,b] \text{ with } |s - t| < \delta.$$

Let $n \in \mathbf{Z}^+$ be such that $\frac{b-a}{n} < \delta$.

Let P be the equally-spaced partition $a = x_0, x_1, \dots, x_n = b$ of $[a,b]$ with

$$x_j - x_{j-1} = \frac{b-a}{n}$$

for each $j = 1, \dots, n$. Then

$$\begin{aligned} U(f,[a,b]) - L(f,[a,b]) &\le U(f,P,[a,b]) - L(f,P,[a,b]) \\ &= \frac{b-a}{n} \sum_{j=1}^{n} \Big(\sup_{[x_{j-1},x_j]} f - \inf_{[x_{j-1},x_j]} f \Big) \\ &\le (b-a)\varepsilon, \end{aligned}$$

where the first equality follows from the definitions of $U(f,[a,b])$ and $L(f,[a,b])$ and the last inequality follows from 1.12.

We have shown that $U(f,[a,b]) - L(f,[a,b]) \le (b-a)\varepsilon$ for all $\varepsilon > 0$. Thus 1.8 implies that $L(f,[a,b]) = U(f,[a,b])$. Hence f is Riemann integrable. ■

An alternative notation for $\int_a^b f$ is $\int_a^b f(x)\,dx$. Here x is a dummy variable, so we could also write $\int_a^b f(t)\,dt$ or use another variable. This notation becomes useful when we want to write something like $\int_0^1 x^2\,dx$ instead of using function notation.

The next result gives a frequently-used estimate for a Riemann integral.

1.13 *Bounds on Riemann integral*

Suppose $f\colon [a,b] \to \mathbf{R}$ is Riemann integrable. Then

$$(b-a)\inf_{[a,b]} f \leq \int_a^b f \leq (b-a)\sup_{[a,b]} f$$

Proof Let P equal the trivial partition $a = x_0, x_1 = b$. Then

$$(b-a)\inf_{[a,b]} f = L(f,P,[a,b]) \leq L(f,[a,b]) = \int_a^b f,$$

proving the first inequality in the result.

The second inequality in the result is proved similarly and is left to the reader. ■

EXERCISES 1A

1 Suppose $f\colon [a,b] \to \mathbf{R}$ is a bounded function such that

$$L(f,P,[a,b]) = U(f,P,[a,b])$$

for some partition P of $[a,b]$. Prove that f is a constant function on $[a,b]$.

2 Suppose $a \leq s < t \leq b$. Define $f\colon [a,b] \to \mathbf{R}$ by

$$f(x) = \begin{cases} 1 & \text{if } s < x < t, \\ 0 & \text{otherwise.} \end{cases}$$

Prove that f is Riemann integrable on $[a,b]$ and that $\int_a^b f = t - s$.

3 Suppose $f\colon [a,b] \to \mathbf{R}$ is a bounded function. Prove that f is Riemann integrable if and only if for each $\varepsilon > 0$, there exists a partition P of $[a,b]$ such that

$$U(f,P,[a,b]) - L(f,P,[a,b]) < \varepsilon.$$

4 Suppose $f,g\colon [a,b] \to \mathbf{R}$ are bounded functions. Prove that

$$L(f,[a,b]) + L(g,[a,b]) \leq L(f+g,[a,b])$$

and

$$U(f+g,[a,b]) \leq U(f,[a,b]) + U(g,[a,b]).$$

5 Suppose $f,g\colon [a,b] \to \mathbf{R}$ are Riemann integrable. Prove that $f+g$ is Riemann integrable on $[a,b]$ and

$$\int_a^b (f+g) = \int_a^b f + \int_a^b g.$$

6 Suppose $f\colon [a,b] \to \mathbf{R}$ is Riemann integrable. Prove that the function $-f$ is Riemann integrable on $[a,b]$ and

$$\int_a^b (-f) = -\int_a^b f.$$

7 Suppose $f\colon [a,b] \to \mathbf{R}$ is Riemann integrable. Suppose $g\colon [a,b] \to \mathbf{R}$ is a function such that $g(x) = f(x)$ for all except finitely many $x \in [a,b]$. Prove that g is Riemann integrable on $[a,b]$ and

$$\int_a^b g = \int_a^b f.$$

8 Suppose $f\colon [a,b] \to \mathbf{R}$ is Riemann integrable. Prove that

$$\int_a^b f = \lim_{n\to\infty} \frac{b-a}{n} \sum_{j=1}^{n} f\bigl(a + \tfrac{j(b-a)}{n}\bigr).$$

9 Suppose $f\colon [a,b] \to \mathbf{R}$ is a bounded function. For $n \in \mathbf{Z}^+$, let P_n denote the partition that divides $[a,b]$ into 2^n intervals of equal size. Prove that

$$L(f,[a,b]) = \lim_{n\to\infty} L(f,P_n,[a,b]) \text{ and } U(f,[a,b]) = \lim_{n\to\infty} U(f,P_n,[a,b]).$$

10 Suppose $f\colon [a,b] \to \mathbf{R}$ is Riemann integrable. Prove that if $c,d \in \mathbf{R}$ and $a \le c < d \le b$, then f is Riemann integrable on $[c,d]$.
[*To say that f is Riemann integrable on $[c,d]$ means that f with its domain restricted to $[c,d]$ is Riemann integrable.*]

11 Suppose $f\colon [a,b] \to \mathbf{R}$ is a bounded function and $c \in (a,b)$. Prove that f is Riemann integrable on $[a,b]$ if and only if f is Riemann integrable on $[a,c]$ and f is Riemann integrable on $[c,b]$. Furthermore, prove that if these conditions hold, then

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

12 Suppose $f\colon [a,b] \to \mathbf{R}$ is Riemann integrable. Define $F\colon [a,b] \to \mathbf{R}$ by

$$F(t) = \begin{cases} 0 & \text{if } t = a, \\ \displaystyle\int_a^t f & \text{if } t \in (a,b]. \end{cases}$$

Prove that F is continuous on $[a,b]$.

13 Suppose $f\colon [a,b] \to \mathbf{R}$ is Riemann integrable. Prove that $|f|$ is Riemann integrable and that

$$\Bigl|\int_a^b f\Bigr| \le \int_a^b |f|.$$

14 Suppose $f\colon [a,b] \to \mathbf{R}$ is a function such that $f(c) \le f(d)$ for all $c,d \in [a,b]$ with $c < d$. Prove that f is Riemann integrable on $[a,b]$.

1B Why the Riemann Integral is Not Good Enough

The Riemann integral works well enough to be taught to millions of calculus students around the world each year. However, the Riemann integral has several deficiencies. In this section, we will discuss the following three issues:

- Riemann integration does not handle functions with many discontinuities;
- Riemann integration does not handle unbounded functions;
- Riemann integration does not work well with limits.

In Chapter 2, we will start to construct a theory to remedy these problems.

We begin with the following example of a function that is not Riemann integrable.

1.14 Example *a function that is not Riemann integrable*

Define $f\colon [0,1] \to \mathbf{R}$ by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

If $[a,b] \subset [0,1]$ with $a < b$, then

$$\inf_{[a,b]} f = 0 \quad \text{and} \quad \sup_{[a,b]} f = 1$$

because $[a,b]$ contains an irrational number (by 0.39) and a rational number (by 0.30). Thus $L(f,P,[0,1]) = 0$ and $U(f,P,[0,1]) = 1$ for every partition P of $[0,1]$. Hence $L(f,[0,1]) = 0$ and $U(f,[0,1]) = 1$. Because $L(f,[0,1]) \neq U(f,[0,1])$, we conclude that f is not Riemann integrable.

This example is disturbing because (as we will see later), there are far fewer rational numbers than irrational numbers. Thus f should, in some sense, have integral 0. However, the Riemann integral of f is not defined.

Trying to apply the definition of the Riemann integral to unbounded functions would lead to undesirable results, as shown in the next example.

1.15 Example *Riemann integration does not work with unbounded functions*

Define $f\colon [0,1] \to \mathbf{R}$ by

$$f(x) = \begin{cases} \frac{1}{\sqrt{x}} & \text{if } 0 < x \leq 1, \\ 0 & \text{if } x = 0. \end{cases}$$

If $x_0, x_1, \dots, x_n$ is a partition of $[0,1]$, then $\sup\limits_{[x_0,x_1]} f = \infty$. Thus if we tried to apply the definition of the upper Riemann sum to f, we would have $U(f,P,[0,1]) = \infty$ for every partition P of $[0,1]$.

However, we should consider the area under the graph of f to be 2, not ∞, because

$$\lim_{a \downarrow 0} \int_a^1 f = \lim_{a \downarrow 0} (2 - 2\sqrt{a}) = 2.$$

Calculus courses deal with the previous example by defining $\int_0^1 \frac{1}{\sqrt{x}}\,dx$ to be $\lim_{a\downarrow 0}\int_a^1 \frac{1}{\sqrt{x}}\,dx$. If using this approach and

$$f(x) = \frac{1}{\sqrt{x}} + \frac{1}{\sqrt{1-x}},$$

then we would define $\int_0^1 f$ to be

$$\lim_{a\downarrow 0}\int_a^{1/2} f + \lim_{b\uparrow 1}\int_{1/2}^b f.$$

However, the idea of taking Riemann integrals over subdomains and then taking limits can fail with more complicated functions, as shown in the next example.

1.16 Example *area seems to make sense, but Riemann integral is not defined*

Let $r_1, r_2, \dots$ be a sequence that includes each rational number in $(0,1)$ exactly once and that includes no other numbers (0.57 implies that such a sequence exists). For $k \in \mathbf{Z}^+$, define $f_k\colon [0,1] \to \mathbf{R}$ by

$$f_k(x) = \begin{cases} \frac{1}{\sqrt{x-r_k}} & \text{if } x > r_k, \\ 0 & \text{if } x \le r_k. \end{cases}$$

Define $f\colon [0,1] \to \mathbf{R}$ by

$$f(x) = \sum_{k=1}^{\infty} \frac{f_k(x)}{2^k}.$$

Because every subinterval of $[0,1]$ with more than one element contains infinitely many rational numbers (as follows from 0.30), f is unbounded on every such subinterval. Thus the Riemann integral of f is undefined on every subinterval of $[0,1]$ with more than one element.

However, the area under the graph of each f_k is less than 2. The formula defining f then shows that we should expect the area under the graph of f to be less than 2 rather than undefined.

The next example shows that the pointwise limit of a sequence of Riemann integrable functions bounded by 1 need not be Riemann integrable.

1.17 Example *Riemann integration does not work well with pointwise limits*

Let $r_1, r_2, \dots$ be a sequence that includes each rational number in $[0,1]$ exactly once and that includes no other numbers (0.57 implies that such a sequence exists). For $k \in \mathbf{Z}^+$, define $f_k\colon [0,1] \to \mathbf{R}$ by

$$f_k(x) = \begin{cases} 1 & \text{if } x \in \{r_1, \dots, r_k\}, \\ 0 & \text{otherwise.} \end{cases}$$

Then each f_k is Riemann integrable and $\int_0^1 f_k = 0$, as you should verify.

Define $f\colon [0,1] \to \mathbf{R}$ by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Clearly

$$\lim_{k \to \infty} f_k(x) = f(x) \quad \text{for each } x \in [0,1].$$

However, f is not Riemann integrable (see Example 1.14) even though f is the pointwise limit of a sequence of integrable functions bounded by 1.

Because analysis relies heavily upon limits, a good theory of integration should allow for interchange of limits and integrals, at least when the functions are appropriately bounded. Thus the previous example points out a serious deficiency in Riemann integration.

Now we come to a positive result, but as we will see, even this result indicates that Riemann integration has some problems.

1.18 ***Interchanging Riemann integral and limit***

Suppose $a, b, M \in \mathbf{R}$ with $a < b$. Suppose $f_1, f_2, \dots$ is a sequence of Riemann integrable functions on $[a,b]$ such that

$$|f_k(x)| \le M$$

for all $k \in \mathbf{Z}^+$ and all $x \in [a,b]$. Suppose $\lim_{k\to\infty} f_k(x)$ exists for each $x \in [a,b]$. Define $f\colon [a,b] \to \mathbf{R}$ by

$$f(x) = \lim_{k \to \infty} f_k(x).$$

If f is Riemann integrable on $[a,b]$, then

$$\int_a^b f = \lim_{k \to \infty} \int_a^b f_k.$$

The result above suffers from two problems. The first problem is the undesirable hypothesis that the limit function f is Riemann integrable. Ideally, that property would follow from the other hypotheses, but Example 1.17 shows that we must explicitly include the assumption that f is Riemann integrable.

The second problem with the result above is that it does not seem to have a reasonable proof using just the tools of Riemann integration. Thus a proof of the result above will not be given here. A proof of a stronger result will be given later, using the tools of measure theory that we will develop starting with the next chapter. The lack of a good Riemann-integration-based proof of the result above indicates that Riemann integration is not the ideal theory of integration.

We have not discussed differentiation (but we will do so in Chapter 4). However, you should recall from your calculus class the following version of the Fundamental Theorem of Calculus: if f is differentiable on an open interval containing $[a,b]$ and f' is continuous on $[a,b]$, then

$$f(b) - f(a) = \int_a^b f'.$$

Note the hypothesis above that f' is continuous on $[a,b]$. It would be nice not to have that hypothesis. However, that hypothesis (or something close to it) is needed because there exist functions f that are differentiable everywhere on an open interval containing $[a,b]$ but f' is not Riemann integrable on $[a,b]$. In other words, if we use Riemann integration, then the right side of the equation above need not make sense even if f' is defined everywhere.

EXERCISES 1B

1 Define $f\colon [0,1] \to \mathbf{R}$ as follows:

$$f(a) = \begin{cases} 0 & \text{if } a \text{ is irrational,} \\ \frac{1}{n} & \text{if } a \text{ is rational and } n \text{ is the smallest positive integer} \\ & \text{such that } a = \frac{m}{n} \text{ for some integer } m. \end{cases}$$

Show that f is Riemann integrable and compute $\int_0^1 f$.

2 Suppose $f\colon [a,b] \to \mathbf{R}$ is a bounded function. Prove that f is Riemann integrable if and only if

$$L(-f,[a,b]) = -L(f,[a,b]).$$

3 Give an example of bounded functions $f, g\colon [0,1] \to \mathbf{R}$ such that

$$L(f+g,[0,1]) \neq L(f,[0,1]) + L(g,[0,1])$$

and

$$U(f+g,[0,1]) \neq U(f,[0,1]) + U(g,[0,1]).$$

4 Give an example of a sequence of continuous real-valued functions $f_1, f_2, \ldots$ on $[0,1]$ and a continuous real-valued function f on $[0,1]$ such that

$$f(x) = \lim_{k\to\infty} f_k(x)$$

for each $x \in [0,1]$ but

$$\int_0^1 f \neq \lim_{n\to\infty} \int_0^1 f_k.$$

5 Show that

$$\lim_{j\to\infty} \Bigl(\lim_{k\to\infty} \bigl(\cos(j!\pi x)\bigr)^{2k} \Bigr) = \begin{cases} 1 & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

for every $x \in \mathbf{R}$.
[*This example is due to Henri Lebesgue.*]